Noncircularity

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Abstract— **This paper examines the second order noncircular properties of Fourier coefficients which are estimated from a time stationary sampled sequence. If $X(m) = (1/\sqrt{M})\sum_{n=0}^{M-1} x(n)\exp(i2\pi mn/M)$, where $x(n)$ is a time stationary data sequence, then the noncircular character of $X(m)$ is shown by $E[X^2(m)] = 2A(m)\csc(2\pi m/M)\exp(i2\pi m/M)$ where $A(m)$ is the sine transform of the autocorrelation function of $x(n)$. The signal processing implications of this are not yet clear, but it appears that it could degrade the performance of a detector by as much as 1.5 dB.**

Keywords— **stationary, circular**

I. Introduction

The real and imaginary parts of a complex random variable, $X = X_R + iX_I$, are often assumed to be zero mean, equal variance, and uncorrelated. In this paper, we shall ignore considerations of nonzero means. In most cases, this will involve no loss of generality. The critical assumption is $E[X_R^2] - E[X_I^2] = E[X_R X_I] = 0$. An equivalent statement is $E[X^2] = 0$. This condition is referred to as second order circularity. Neeser and Massey [1] describe such variables as "proper." In the Gaussian case this leads to the conclusion that the probability density function depends only on the magnitude of the variable. Picinbono [2] has recently clarified some of the issues associated with circularity and described other types of circularity. This paper will confine itself to second order issues. Noncircular random variables have received less attention but may occur more often. They represent the general case and deserve study.

The variables of interest in this paper will be the coefficients which result from discrete Fourier transforms of time stationary data. Such coefficients often arise in the study of time series data for spectrum analysis and sensor array processing. They are often assumed to approximate the coefficients of the spectral representation and therefore to be circular. Since circularity substantially simplifies many processors it is disappointing to find that these coefficients are not circular.

II. The Circular Anomaly

To make the problem more specific, let $x(n)$ represent a time stationary data sequence with an autocorrelation function

$$R(k) = E[x(n+k)x(n)] \tag{1}$$

We shall use M sequential samples to form a Fourier transform coefficient in the usual manner. The real and imaginary parts of the estimate are

$$X_R(m) = \frac{1}{\sqrt{f_s M}} \sum_{n=0}^{M-1} x(n)\cos\left(\frac{2\pi mn}{M}\right) \tag{2}$$

$$X_I(m) = -\frac{1}{\sqrt{f_s M}} \sum_{n=0}^{M-1} x(n)\sin\left(\frac{2\pi mn}{M}\right) \tag{3}$$

Let us first examine the correlation between $X_R(m)$ and $X_I(m)$.

$$E[X_R(m)X_I(m)] = \tag{4}$$

$$\frac{-1}{f_s M}\sum_{n=0}^{M-1}\sum_{k=0}^{M-1} E[x(n)x(k)]\cos\left(\frac{2\pi mn}{M}\right)\sin\left(\frac{2\pi km}{M}\right)$$

$$= \frac{-1}{2f_s M}\sum_{n=0}^{M-1}\sum_{k=0}^{M-1} R(n-k)\sin\left(\frac{2\pi(n+k)m}{M}\right)$$

$$+\frac{-1}{2f_s M}\sum_{n=0}^{M-1}\sum_{k=0}^{M-1} R(n-k)\sin\left(\frac{2\pi(k-n)m}{M}\right)$$

From symmetry, the second term in the last expression will sum to zero. The first term is

$$\frac{-1}{2f_s M}\sum_{n=0}^{M-1}\sum_{k=0}^{M-1} R(n-k)\sin\left(\frac{2\pi(n+k)m}{M}\right)$$

$$= \frac{-1}{f_s M}\sum_{n=1}^{M-1}\sum_{k=0}^{M-1-n} R(n)\sin\left(\frac{2\pi(n+2k)m}{M}\right)$$

$$E[X_R(m)X_I(m)] = \frac{-1}{f_s M}\sum_{n=1}^{M-1} R(n)\sin\left(\frac{2\pi mn}{M}\right) \tag{5}$$

To get the variance of $X_R(m)$,

$$E[X_R^2(m)] =$$

$$\frac{1}{f_s M}\sum_{n=0}^{M-1}\sum_{k=0}^{M-1} E[x(n)x(k)]\cos\left(\frac{2\pi mn}{M}\right)\cos\left(\frac{2\pi km}{M}\right) \tag{6}$$

while for $X_I(m)$

$$E[X_I^2(m)] =$$

$$\frac{1}{f_s M}\sum_{n=0}^{M-1}\sum_{k=0}^{M-1} E[x(n)x(k)]\sin\left(\frac{2\pi mn}{M}\right)\sin\left(\frac{2\pi km}{M}\right) \tag{7}$$

Adding these two equations gives the spectral estimate

$$S(m) = E[X_R^2(m)] + E[X_I^2(m)] =$$

$$\frac{1}{f_s M}\sum_{n=0}^{M-1}\sum_{k=0}^{M-1} R(n-k)\cos\left(\frac{2\pi(n-k)m}{M}\right) \tag{8}$$

EDICS no. SPL.SP.3.5

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The author has received valuable comments and assistance from Prof. Don H. Johnson of Rice University.

while the difference is

$$(E[X_R^2(m)] - E[X_I^2(m)])(f_s M) =$$

$$\sum_{n=0}^{M-1}\sum_{k=0}^{M-1} R(n-k)\cos\left(\frac{2\pi(n+k)m}{M}\right)$$

$$= \sum_{n=0}^{M-1} R(0)\cos\left(\frac{4\pi mn}{M}\right) +$$

$$2\sum_{n=1}^{M-1} R(n) \sum_{k=0}^{M-1-n} \cos\left(\frac{2\pi(2k+n)m}{M}\right)$$

$$E[X_R^2(m)] - E[X_I^2(m)] =$$

$$\frac{-2}{f_s M}\cot\left(\frac{2\pi m}{M}\right)\sum_{n=1}^{M-1} R(n)\sin\left(\frac{2\pi mn}{M}\right) \tag{9}$$

This suggests that we define a new quantity which we shall call the circular anomaly,

$$A(m) = \frac{-1}{f_s M}\sum_{n=1}^{M-1} R(n)\sin\left(\frac{2\pi mn}{M}\right) \tag{10}$$

This will lead to

$$E[X_R(m)X_I(m)] = A(m) \tag{11}$$

$$E[X_R^2(m)] - E[X_I^2(m)] = 2\cot\left(\frac{2\pi m}{M}\right)A(m) \tag{12}$$

In general, it is not possible to characterize the noncircular behavior of a variable by a single real number, because the phase of $E[X^2]$ is also important. However, under the time stationarity assumption above, the phase is shown below to be knowable *a priori*.

If we deal with a continuous time sequence, the above summations will be replaced with integrals for a Fourier series expansion. In that case the anomaly will be zero. Therefore, the circularity is lost when the data are sampled. To establish how the continuous case is a limit of the discrete case as M increases, it is evidently necessary to make M/f_s constant.

III. Implications of Stationary

Equations 11 and 12 show the form which the second order moments of $X(m)$ must have if $X(m)$ corresponds to a stationary data sequence. Unless the noise is white, the circular anomaly will usually not be zero, and the second order statistics must take a prescribed form which is not circular.

The implications of this anomaly have evidently never been effectively studied. Most of them are beyond the scope of this paper. However, we can make a few interesting observations. The real covariance matrix of the real and imaginary components takes the form

$$\begin{bmatrix} \frac{S(m)}{2} + \cot\left(\frac{2\pi m}{M}\right)A(m) & A(m) \\ A(m) & \frac{S(m)}{2} - \cot\left(\frac{2\pi m}{M}\right)A(m) \end{bmatrix} \tag{13}$$

Eigenvalues of this matrix are $S(m)/2+A(m)\csc(2\pi m/M)$ and $S(m)/2-A(m)\csc(2\pi m/M)$. More remarkable are the eigenvectors of this matrix, which are

$$\begin{bmatrix}\cos(\pi m/M)\\ \sin(\pi m/M)\end{bmatrix}, \begin{bmatrix}-\sin(\pi m/M)\\ \cos(\pi m/M)\end{bmatrix} \tag{14}$$

In other words, they are independent of the data. If one wished to design a diagonalizing transformation for the real and imaginary parts, he could do so from *a priori* information. The transformation will take the simple form $Y(m) = X(m)\exp(-i\pi m/M)$. $Y(m)$ will then correspond to a time sequence which is delayed $1/(2f_s)$ from the original sequence. This will, of course, maximize the difference in magnitude between the components. The condition number of the matrix in either case will be

$$\frac{S(m) + 2A(m)\csc(2\pi m/M)}{S(m) - 2A(m)\csc(2\pi m/M)} \tag{15}$$

or the reciprocal of this, depending on the sign of $A(m)$. This condition number is important because it indicates the extent to which one component of $X(m)$ dominates the other. For example, in the Gaussian case if the condition number is very large, $X(m)X^*(m)$ would approximate a chi-squared one degree of freedom variable instead of the two degrees of freedom which would occur if the condition number was one.

As mentioned earlier, second order circularity is sometimes associated with the requirement that $E[X^2] = 0$. We can easily write the value to be expected in the time stationary case. Since

$$E[X^2(m)] = 2A(m)(\cot(2\pi m/M) + i)$$

$$E[X^2(m)] = \frac{2A(m)}{\sin(2\pi m/M)}\exp(i2\pi m/M) \tag{16}$$

A test for this phase angle could, in principle, be used as a test for stationarity.

The signal processing implications of this are not yet clear. The most obvious implication is that for Gaussian noise XX^* will not be an exponential variable, but rather a sum of unequal chi-square variables. In an extreme case, (large condition number) it would result in an effective loss of half of the supposed number of degrees of freedom in the estimator, presumably with a 1.5 dB loss in detection performance.

References

[1] Fredy D Neeser and James L. Massey, "Proper complex random processes with applications to information theory", *IEEE Transactions on Information Theory*, vol. 39, no. 4, pp. 1293–1302, July 1993.

[2] Bernard Picinbono, "On circularity", *IEEE Transactions on Signal Processing*, vol. 42, no. 12, pp. 3473–3482, Dec. 1994.