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Modeling of Membrane Waves

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EXECUTIVE SUMMARY

OBJECTIVE

The objective of this work was to examine the theoretical formulation of shell membrane waves to determine how a computationally efficient, yet faithful, model could be realized.

RESULTS

Faithful replication of the behavior may be obtained from a relatively simple set of expressions. These produce tremendous computational savings.

RECOMMENDATIONS

The proposed model should be implemented as soon as feasible.

CONTENTS

Figures

Figures (contd)

INTRODUCTION

Accurate target models capable of predicting the echo time history of submarines and surface ships are essential tools for antisubmarine warfare and ship vulnerability studies. A realistic model must account for the many highlights of a target as well as the conditions of insonification. The model must predict the proper highlight amplitude and phase as well as temporal and spectral behavior. Thus, physical scattering mechanisms must be properly identified and incorporated into the model.

Over the past 30 years, target models developed at Space and Naval Warfare (SPAWAR) Systems Center, San Diego (SSC San Diego) have proven to be an invaluable asset for developing signal-processing algorithms and engagement simulations. These models have been exported throughout the Navy community. Previous formulations of the SSC San Diego target models, dictated by past requirements and test scenarios, have neglected elastic wave effects. This restriction simplified the numerical implementation (as well as theoretical development) considerably. This restriction serves well for high-frequency and even narrowband mid-frequency applications. However, for broadband processing schemes, more sophistication is required.

Efforts have been undertaken, e.g., interacting ribs (Lengua, 1997). This report examines the modeling of shell membrane waves. We will begin with general background information since many interested parties are unfamiliar with the subject. Then we will discuss the modeling details.

BACKGROUND

MEMBRANE SHELL THEORY

We will take the approach of beginning with the simplest shell theory and then proceed with more complicated formulations. The theory discussed in this section is known as membrane shell theory. We will follow the development of Graff (1975, pp. 259–262). Only forces, both normal and shear, acting in the midsurface of the shell are considered. The transverse shear forces and the bending and twisting moments are assumed negligible. Thus, the shell behaves as a curved membrane.

Consider a cylindrical shell of radius a, thickness h, and density ρ. We will use cylindrical coordinates (r, θ, z). The displacement components are w, v, and u in the radial, tangential, and longitudinal directions. Let us examine the forces (membrane stresses N and applied load q) acting on a differential element of the shell. The equations of motion in the longitudinal, tangential, and radial directions are, respectively,

$$-N_z a d\theta + \left(N_z + \frac{\partial N_z}{\partial z} dz\right) a d\theta - N_{\theta z} dz + \left(N_{\theta z} + \frac{\partial N_{\theta z}}{\partial \theta}\right) dz = \rho a d\theta dz h \frac{\partial^2 u}{\partial t^2}$$

$$-N_\theta dz + \left(N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta\right) dz - N_{z\theta} a d\theta + \left(N_{z\theta} + \frac{\partial N_{z\theta}}{\partial z}\right) a d\theta = \rho a d\theta dz h \frac{\partial^2 v}{\partial t^2}$$

$$-N_\theta \frac{d\theta}{2} dz - \left(N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta \right) \frac{d\theta}{2} dz + qad\theta dz = \rho ad\theta dzh \frac{\partial^2 w}{\partial t^2} \quad .$$

These may be reduced to

$$\frac{\partial N_z}{\partial z} + \frac{1}{a} \frac{\partial N_{\theta z}}{\partial \theta} = \rho h \frac{\partial^2 u}{\partial t^2}$$

$$\frac{1}{a} \frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{z\theta}}{\partial z} = \rho h \frac{\partial^2 v}{\partial t^2}$$

$$-\frac{N_\theta}{a} + q = \rho h \frac{\partial^2 w}{\partial t^2} .$$

The membrane stresses $(N_z, N_\theta, N_{\theta z}, N_{z\theta})$ are obtained by integrating the usual stresses $(\Sigma_z, \Sigma_\theta, \tau_{\theta z}, \tau_{z\theta})$ across the shell thickness. From Hooke's law, we have

$$\Sigma_z = \frac{E}{1-\sigma^2} (\varepsilon_z + \sigma \varepsilon_\theta)$$

$$\Sigma_\theta = \frac{E}{1-\sigma^2} (\varepsilon_\theta + \sigma \varepsilon_z)$$

$$\tau_{\theta z} = \tau_{z\theta} = G\gamma = \frac{E}{2(1+\sigma)} \gamma$$

where ε_z and ε_θ are the axial and tangential strains of the middle surface of the shell element and $\gamma = \gamma_{\theta z} = \gamma_{z\theta}$ is the shear strain. E is Young's modulus and σ is Poisson's ratio. For the assumption of membrane-type stresses only, $(\Sigma_z, \Sigma_\theta, \tau_{\theta z}, \tau_{z\theta})$ are constant across the shell thickness. So then

$$N_z = \frac{Eh}{1-\sigma^2} (\varepsilon_z + \sigma \varepsilon_\theta)$$

$$N_\theta = \frac{Eh}{1-\sigma^2} (\varepsilon_\theta + \sigma \varepsilon_z)$$

$$N_{\theta z} = N_{z\theta} = \frac{Eh}{2(1+\sigma)} \gamma \quad .$$

Let us now consider the dynamics of the deformation. For the present conditions, these are relatively simple. In the axial direction

$$\varepsilon_z = \frac{\partial u}{\partial z} .$$

Now $\varepsilon_\theta = \dfrac{ds' - ds}{ds}$, where $ds = ad\theta$ is the initial arc length. The arc length after deformation is $ds' = (w + a)d\theta + \dfrac{\partial v}{\partial \theta} d\theta$. Therefore

$$\varepsilon_\theta = \frac{1}{a}\left(w + \frac{\partial v}{\partial \theta} \right).$$

The expression for the shear strain may be obtained by considering small changes in angle of the sides dz and $ad\theta$ of the element due to $\dfrac{\partial v}{\partial z}$ and $\dfrac{\partial u}{\partial \theta}$. The result is

$$\gamma = \frac{\partial v}{\partial z} + \frac{1}{a}\frac{\partial u}{\partial \theta}.$$

The membrane stresses are then

$$N_z = \frac{Eh}{1-o^2}\left[\frac{\partial u}{\partial z} + \frac{\sigma}{a}\left(w + \frac{\partial v}{\partial \theta} \right) \right]$$

$$N_\theta = \frac{Eh}{1-o^2}\left[\frac{w}{a} + \frac{1}{a}\frac{\partial v}{\partial \theta} + \sigma \frac{\partial u}{\partial z} \right]$$

$$N_{\theta z} = N_{z\theta} = \frac{Eh}{1-\sigma^2}\left(\frac{1-\sigma}{2} \right)\left[\frac{\partial v}{\partial z} + \frac{1}{a}\frac{\partial u}{\partial \theta} \right].$$

The equations of motion may finally be written as

$$\frac{\partial^2 u}{\partial z^2} + \left(\frac{1-\sigma}{2a^2} \right)\frac{\partial^2 u}{\partial \theta^2} + \left(\frac{1+\sigma}{2a} \right)\frac{\partial^2 v}{\partial z \partial \theta} + \left(\frac{\sigma}{a} \right)\frac{\partial w}{\partial z} = \frac{1}{c_p^2}\frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{1-\sigma}{2} \right)\frac{\partial^2 v}{\partial z^2} + \left(\frac{1}{a^2} \right)\frac{\partial^2 v}{\partial \theta^2} + \left(\frac{1+\sigma}{2a} \right)\frac{\partial^2 u}{\partial z \partial \theta} + \left(\frac{1}{a^2} \right)\frac{\partial w}{\partial \theta} = \frac{1}{c_p^2}\frac{\partial^2 v}{\partial t^2}$$

$$-\frac{w}{a^2} - \left(\frac{\sigma}{a} \right)\frac{\partial u}{\partial z} - \left(\frac{1}{a^2} \right)\frac{\partial v}{\partial \theta} + \frac{1-\sigma^2}{Eh} q = \frac{1}{c_p^2}\frac{\partial^2 w}{\partial t^2}$$

where $c_p = \sqrt{\dfrac{E}{\rho(1-\sigma^2)}}$ is the thin-plate speed.

From this last equation, we see that the normal displacement of the shell, w, is coupled to the two “in-plane” displacements, u and v. For flat plates, the normal displacement is independent of the in-plane displacements.

DONNELL SHELL THEORY

Formulation

The analysis of a shell including bending effects on the deformation, as well as bending moments and shear forces in the equations of motion, yields considerably more complex expressions than those in the preceding section. The Donnell formulation (Kraus, 1967, p. 297) includes simplified versions of these effects. The approximations are related to the influence of transverse shear forces on tangential motion and to the expressions for curvature and twist. The results are equations very much like the membrane equations of motion. There is only an additional term of $-\frac{h^2}{12}\nabla^4 w$ on the left side of the radial displacement equation

$$-\frac{w}{a^2}-\beta^2\left[a^2\frac{\partial^4 w}{\partial z^4}+2\frac{\partial^4 w}{\partial z^2\partial\theta^2}+\frac{1}{a^2}\frac{\partial^4 w}{\partial\theta^4}\right]-\left(\frac{\sigma}{a}\right)\frac{\partial u}{\partial z}-\left(\frac{1}{a^2}\right)\frac{\partial v}{\partial\theta}+\frac{1-\sigma^2}{Eh}q=\frac{1}{c_p^2}\frac{\partial^2 w}{\partial t^2}$$

where $\beta^2=\frac{h^2}{12a^2}$.

We wish to study the propagation of harmonic waves in the shell. These will have an angular frequency ω and wavenumber k . Note that, since we are using a “thin-shell” analysis, it is required that $kh << \pi$, or equivalently $\beta ka << 1$. For high frequencies, a thick-plate analysis will be necessary (Junger and Feit, 1986, pp. 214–215). Rather than begin with the general solution, it is illustrative to consider some special cases.

Case of θ-Independent Motion

One of the most important special cases arises from considering motion independent of θ . Here the equations of motion are (also set $q=0$)

$$\frac{\partial^2 u}{\partial z^2}+\left(\frac{\sigma}{a}\right)\frac{\partial w}{\partial z}=\frac{1}{c_p^2}\frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{1-\sigma}{2}\right)\frac{\partial^2 v}{\partial z^2}=\frac{1}{c_p^2}\frac{\partial^2 v}{\partial t^2}$$

$$-\frac{w}{a^2}-\beta^2 a^2\frac{\partial^4 w}{\partial z^4}-\left(\frac{\sigma}{a}\right)\frac{\partial u}{\partial z}=\frac{1}{c_p^2}\frac{\partial^2 w}{\partial t^2} \quad .$$

Note that the equation for tangential motion has uncoupled from the remaining equations. It may be written as

$$\frac{\partial^2 v}{\partial z^2}=\frac{1}{c_s^2}\frac{\partial^2 v}{\partial t^2}$$

where $c_s = \sqrt{\dfrac{G}{\rho}}$ is the shear speed. This is a one-dimensional equation describing the purely torsional motion of the shell. The propagation speed is the same as for such waves in a solid circular rod (Graff, 1975, p. 263).

Consider now the coupled equations in u and w. Let $u = Ue^{i(kz-\omega t)}$ and $w = We^{i(kz-\omega t)}$. A solution requires

$$\begin{bmatrix} \dfrac{\omega^2}{c_p^2} - k^2 & ik\dfrac{\sigma}{a} \\ -ik\dfrac{\sigma}{a} & \dfrac{\omega^2}{c_p^2} - \dfrac{1}{a^2} - \beta^2 a^2 k^4 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix} = 0 \quad .$$

The determinant of coefficients gives the frequency equation

$$\Omega^4 - \left[1 + (ka)^2 + \beta^2 (ka)^4\right]\Omega^2 + \left[1 - \sigma^2 + \beta^2 (ka)^4\right](ka)^2 = 0$$

where $\Omega = \dfrac{a}{c_p}\omega$ is the normalized frequency. In terms of the phase velocity $c = \dfrac{\omega}{k}$, we have

$$\left(\frac{c}{c_p}\right)^4 - \left[1 + \frac{1}{(ka)^2} + \beta^2 (ka)^2\right]\left(\frac{c}{c_p}\right)^2 + \frac{1-\sigma^2}{(ka)^2} + \beta^2 (ka)^2 = 0 \quad .$$

The long- and short-wavelength limits are easily obtained. At long wavelengths ($ka \to 0$)

$$(ka)^2 \left(\frac{c}{c_p}\right)^4 - \left(\frac{c}{c_p}\right)^2 + 1 - \sigma^2 = 0$$

so one solution is $c \to \sqrt{\dfrac{E}{\rho}} = c_b$, the longitudinal bar speed (Graff, 1975, p. 264). In this case, $\left|\dfrac{W}{U}\right| \propto ka$, so the motion is primarily longitudinal (as expected). Another solution is $c \to \dfrac{c_p}{ka}$. Here $\left|\dfrac{U}{W}\right| \propto ka$, the motion is primarily radial.

At short wavelengths ($ka \to \infty$)

$$\frac{1}{(ka)^2}\left(\frac{c}{c_p}\right)^4 - \beta^2 \left(\frac{c}{c_p}\right)^2 + \beta^2 = 0$$

so one solution is $c \to c_p$. Then $W = 0$, the motion is primarily longitudinal (quasi-compressional wave). Another solution is $c \to \beta kac_p$. Here $\left|\frac{U}{W}\right| \propto \frac{1}{ka}$, the motion is primarily radial (quasi-flexural wave).

Figure 1 shows the dispersion curves when $\sigma = 0.3$ and $\beta = 0.01$. Figure 2 shows $|U|$ for the two branches (normalized such that $|U|^2 + |W|^2 = 1$).

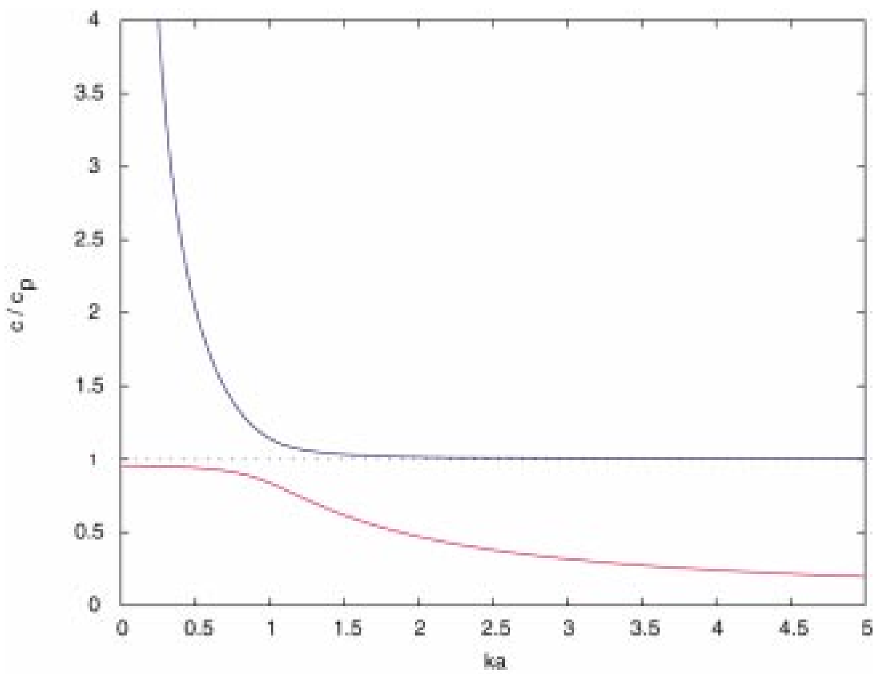

Figure 1. Example of dispersion curves for θ-independent motion.

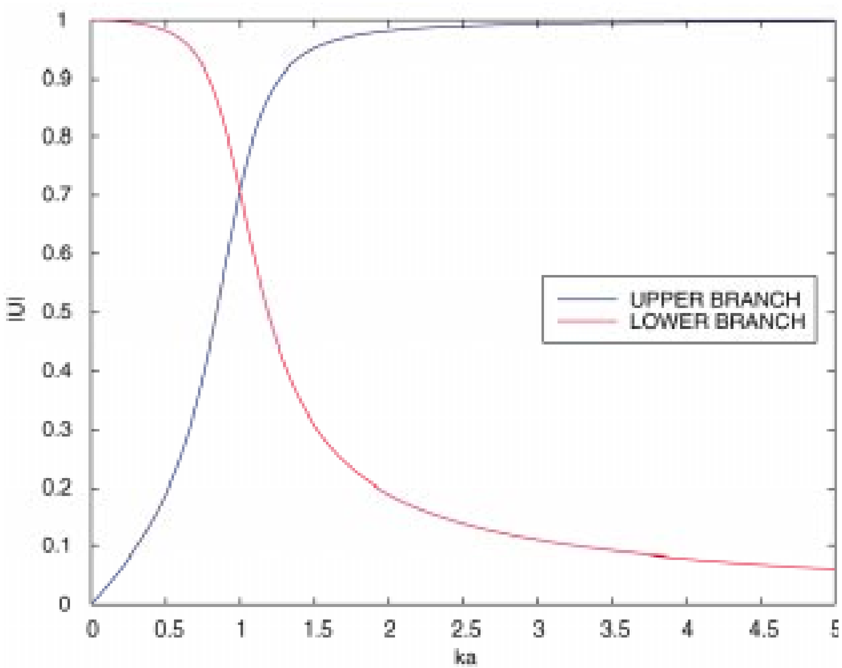

Figure 2. Example of longitudinal displacement for θ-independent motion.

Case of z-Independent Motion

Another important special case arises from considering motion independent of z. Here the equations of motion are (also set $q = 0$)

$$\left(\frac{1-\sigma}{2a^2}\right)\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c_p^2}\frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{1}{a^2}\right)\frac{\partial^2 v}{\partial \theta^2} + \left(\frac{1}{a^2}\right)\frac{\partial w}{\partial \theta} = \frac{1}{c_p^2}\frac{\partial^2 v}{\partial t^2}$$

$$-\frac{w}{a^2} - \beta^2 \frac{1}{a^2}\frac{\partial^4 w}{\partial \theta^4} - \left(\frac{1}{a^2}\right)\frac{\partial v}{\partial \theta} = \frac{1}{c_p^2}\frac{\partial^2 w}{\partial t^2} \quad .$$

Note that the equation for longitudinal motion has uncoupled from the remaining equations. It may be written as

$$\frac{\partial^2 u}{\partial (a\theta)^2} = \frac{1}{c_s^2}\frac{\partial^2 u}{\partial t^2}$$

so that again disturbances propagate at the shear speed.

Consider now the coupled equations in v and w. Let $v = Ve^{i(\kappa a\theta - \omega t)}$ and $w = We^{i(\kappa a\theta - \omega t)}$. Now we must have our first discussion of boundary conditions. These are namely continuity of the displacements for θ and $\theta + 2\pi$

$$v(\theta + 2\pi) = v(\theta)$$

and similarly for w. Thus $\kappa a = n$, where n is an integer (≥ 0). Therefore, only discrete modes exist. Note that we must have $\beta n << 1$ for the development to be valid.

A solution requires

$$\begin{bmatrix} \dfrac{\omega^2}{c_p^2} - \kappa^2 & i\kappa \dfrac{1}{a} \\ -i\kappa \dfrac{\sigma}{a} & \dfrac{\omega^2}{c_p^2} - \dfrac{1}{a^2} - \beta^2 a^2 \kappa^4 \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} = 0 \quad .$$

The determinant of coefficients gives the frequency equation

$$\Omega^4 - \left[1 + n^2 + \beta^2 n^4\right]\Omega^2 + \beta^2 n^6 = 0 \quad .$$

In terms of the phase velocity

$$\left(\frac{c}{c_p}\right)^4 - \left[1 + \frac{1}{n^2} + \beta^2 n^2\right]\left(\frac{c}{c_p}\right)^2 + \beta^2 n^2 = 0 \quad .$$

For the $n = 0$ mode, one solution is $c \to 0$. Here $\omega = 0$ and $W = 0$, so it is a trivial solution. Another solution is $c \to \infty$. In this case, $\omega = \dfrac{c_p}{a}$ and $V = 0$, which is a pulsating cylinder.

At short wavelengths ($n \to \infty$), one solution is $c \to c_p$. Then $W = 0$, the motion is primarily tangential (quasi-compressional wave). Another solution is $c \to \beta n c_p$. Here, $\left|\dfrac{V}{W}\right| \propto \dfrac{1}{n}$, the motion is primarily radial (quasi-flexural wave).

Figure 3 shows the dispersion curves when $\beta = 0.01$. These are plotted as continuous functions of n as a guide. Figure 4 shows $|V|$ for the two branches (normalized such that $|V|^2 + |W|^2 = 1$).

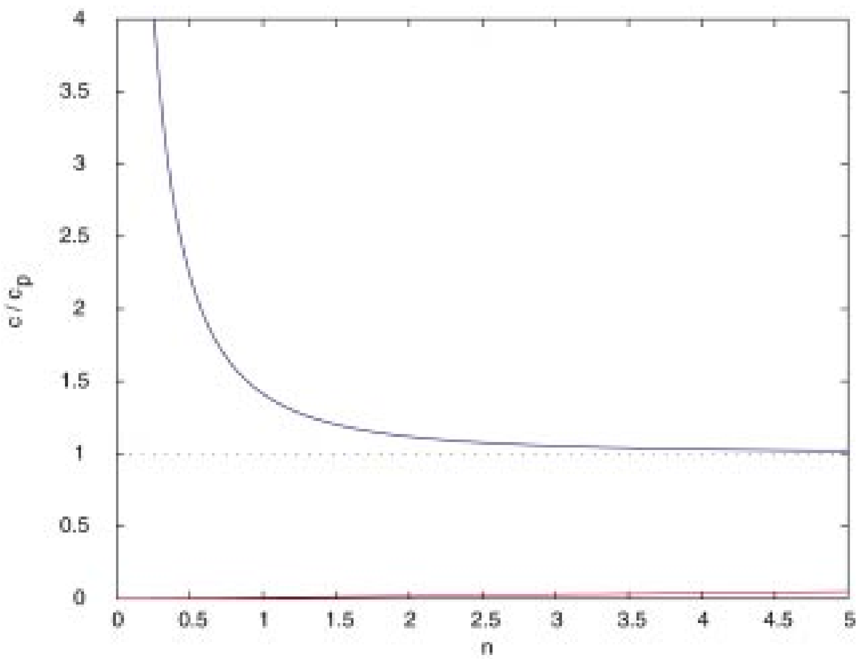

Figure 3. Example of dispersion curves for z-independent motion.

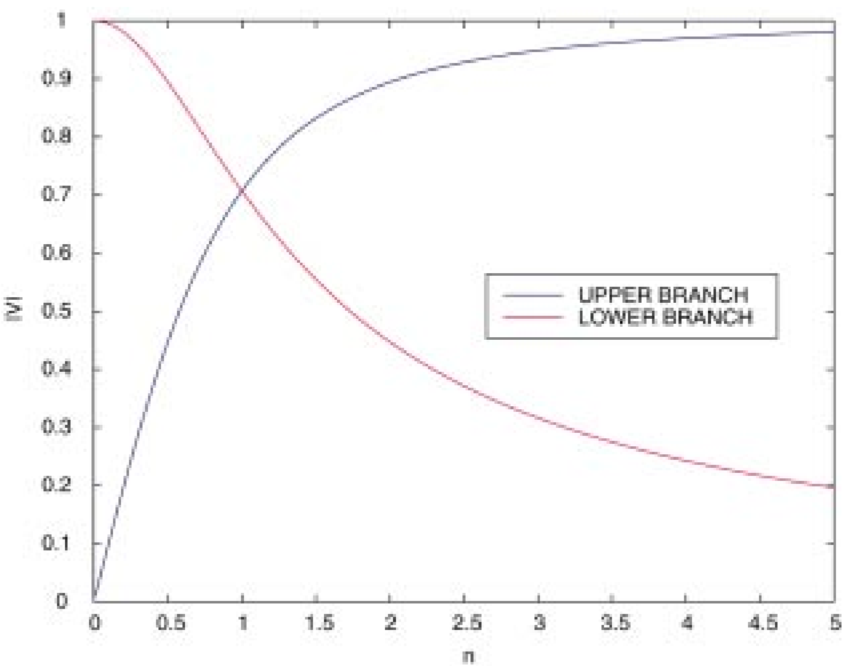

Figure 4. Example of tangential displacement for z–independent motion.

Forced Planar Vibrations

Let us again consider z-independent motion but now with an applied load that is symmetric in θ .

$$q = f(\theta)e^{-i\omega t} \quad .$$

We may represent $f(\theta)$ as a Fourier expansion

$$f(\theta) = \sum_{n=0}^{\infty} f_n \cos(n\theta)$$

where

$$f_n = \frac{\varepsilon_n}{\pi}\int_0^{\pi} f(\phi)\cos(n\phi)d\phi$$

with $\varepsilon_0 = 1$ and $\varepsilon_n = 2$ for $n \geq 1$.

We will assume the displacements are

$$u = 0$$

$$v = \sum_{n=1}^{\infty} V_n \sin(n\theta)e^{-i\omega t}$$

$$w = \sum_{n=1}^{\infty} W_n \cos(n\theta)e^{-i\omega t} \quad .$$

The equations of motion, after using the trigonometric orthogonality relations

$$\frac{\varepsilon_n}{2\pi}\int_{-\pi}^{\pi} \cos(m\phi)\cos(n\phi)d\phi = \delta_{mn}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(m\phi)\sin(n\phi)d\phi = \delta_{mn} \quad ,$$

are then

$$\begin{bmatrix} \Omega^2 - n^2 & -n \\ -n & \Omega^2 - 1 - \beta^2 n^4 \end{bmatrix} \begin{bmatrix} V_n \\ W_n \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{f_n a^2}{\rho h c_p^2} \end{bmatrix} \quad .$$

Thus,

$$v = -\frac{a^2}{\rho h c_p^2}\sum_{n=1}^{\infty}\frac{n f_n \sin(n\theta) e^{-i\omega t}}{\Omega^4 - \left(1 + n^2 + \beta^2 n^4\right)\Omega^2 + \beta^2 n^6}$$

$$w = -\frac{a^2}{\rho h c_p^2}\sum_{n=0}^{\infty}\frac{\left(\Omega^2 - n^2\right) f_n \cos(n\theta) e^{-i\omega t}}{\Omega^4 - \left(1 + n^2 + \beta^2 n^4\right)\Omega^2 + \beta^2 n^6} \quad .$$

These are divergent at the modal frequencies because dissipation has been ignored.

Later it will be convenient to express the response in terms of the modal mechanical impedance

$$Z_n^s(\Omega) = -\frac{i\rho c_p}{\Omega}\frac{h}{a}\frac{\Omega^4 - \left(1 + n^2 + \beta^2 n^4\right)\Omega^2 + \beta^2 n^6}{\Omega^2 - n^2} \quad .$$

Therefore,

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty}\frac{n f_n \sin(n\theta) e^{-i\omega t}}{\left(\Omega^2 - n^2\right) Z_n^s}$$

$$\frac{\partial w}{\partial t} = \sum_{n=0}^{\infty}\frac{f_n \cos(n\theta) e^{-i\omega t}}{Z_n^s}$$

so that Z_n^s is the ratio of the modal pressure on the outer shell surface to the modal radial velocity.

Note that we have been considering the *in vacuo* vibrations of the shell. A shell vibrating in a fluid will radiate into the fluid. The radiation loading is significant and will be discussed later.

General Motion

In considering general motion, let

$$u = U e^{i(n\theta + kz - \omega t)}$$

$$v = V e^{i(n\theta + kz - \omega t)}$$

$$w = W e^{i(n\theta + kz - \omega t)} \quad .$$

Solution of the equations of motion (with $q = 0$) requires

$$\begin{bmatrix} \frac{\varpi^2}{c_p^2} - k^2 - \frac{1-\sigma}{2a^2} n^2 & -\frac{1+\sigma}{2a} nk & i\frac{\sigma}{a}k \\ -\frac{1+\sigma}{2a} nk & \frac{\varpi^2}{c_p^2} - \frac{1-\sigma}{2} k^2 - \frac{n^2}{a^2} & i\frac{n}{a^2} \\ -i\frac{\sigma}{a}k & -i\frac{n}{a^2} & \frac{\varpi^2}{c_p^2} - \frac{1}{a^2} - \frac{\beta^2}{a^2}\left(n^2 + k^2a^2\right)^2 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = 0 \quad .$$

The determinant of coefficients gives the frequency equation

$$\Omega^6 - A_2\Omega^4 + A_1\Omega^2 - A_0 = 0$$

where the parameters are

$$A_2 = 1 + \frac{3-\sigma}{2}(k_s a)^2 + \beta^2 (k_s a)^4$$

$$A_1 = \frac{1-\sigma}{2}\left[(k_s a)^4 + (k_s a)^2 + 2(1+\sigma)(ka)^2\right] + \frac{3-\sigma}{2}\beta^2 (k_s a)^6$$

$$A_0 = \frac{1-\sigma}{2}\left[(1-\sigma^2)(ka)^4 + \beta^2 (k_s a)^8\right] \ ,$$

and we have defined a helical wavenumber

$$k_s = \sqrt{k^2 + \frac{n^2}{a^2}} \quad .$$

The phase velocity is then $\frac{c}{c_p} = \frac{\Omega}{k_s a}$. Here, we require $\beta k_s a << 1$ for the development to be valid.

For the $n = 0$ mode, we retrieve the θ-independent results. Similarly, if $k = 0$, we retrieve the z-independent results.

Let us consider the situation when $k_s a >> 1$. The phase velocity equation may be written as

$$\left(\frac{c}{c_p}\right)^6 - \frac{3-\sigma}{2}\left(\frac{c}{c_p}\right)^4 + \frac{1-\sigma}{2}\left(\frac{c}{c_p}\right)^2 - \frac{1-\sigma}{2}\beta^2 (k_s a)^2 = 0 \quad .$$

The solutions are then $c \to c_p$, $c \to \beta k_s a c_p$, and $c \to c_s$. To examine the displacements, we must know the relative contributions of ka and n. Let us first suppose $ka >> n$. For $c \to c_p$, $\left|\frac{V}{U}\right| \propto \frac{n}{ka}$ and $\left|\frac{W}{U}\right| \propto \frac{n^2}{ka}$, the motion is primarily longitudinal (quasi-compressional wave). For $c \to \beta kac_p$, $\left|\frac{U}{W}\right| \propto \frac{1}{ka}$ and $\left|\frac{V}{W}\right| \propto \frac{n}{(ka)^2}$, and the motion is primarily radial (quasi-flexural wave). For $c \to c_s$, $\left|\frac{U}{V}\right| \propto \frac{n}{ka}$ and $\left|\frac{W}{V}\right| \propto \frac{n}{(ka)^2}$, so the motion is primarily tangential. Now suppose $n >> ka$. For $c \to c_p$, $\left|\frac{U}{V}\right| \propto \frac{ka}{n}$ and $\left|\frac{W}{V}\right| \propto \frac{(ka)^2}{n}$, the motion is primarily tangential (quasi-compressional wave). For $c \to \beta nc_p$, $\left|\frac{U}{W}\right| \propto \frac{ka}{n^2}$ and $\left|\frac{V}{W}\right| \propto \frac{1}{n}$, and the motion is again primarily radial (quasi-flexural wave). For $c \to c_s$, $\left|\frac{V}{U}\right| \propto \frac{ka}{n}$ and $\left|\frac{W}{U}\right| \propto \frac{ka}{n^2}$, the motion is primarily longitudinal.

The frequency equation is a cubic equation in Ω^2 and may therefore be solved using the standard method (Gautschi, 1965, p. 17)]. Let

$$Q = \frac{1}{3}A_1 - \frac{1}{9}A_2^2$$

$$R = -\frac{1}{6}(A_1A_2 - 3A_0) + \frac{1}{27}A_2^3 \quad .$$

If $Q^3 + R^2 > 0$, there is one real root and a complex conjugate pair. If $Q^3 + R^2 = 0$, all roots are real and at least two are equal. If $Q^3 + R^2 < 0$, all roots are real and distinct. Let

$$S_{1,2} = \left[R \pm (Q^3 + R^2)^{1/2}\right]^{1/3} \quad .$$

Then

$$\Omega_1^2 = S_1 + S_2 + \frac{1}{3}A_2$$

$$\Omega_2^2 = -\frac{1}{2}(S_1 + S_2) + \frac{1}{3}A_2 + \frac{\sqrt{3}}{2}i(S_1 - S_2)$$

$$\Omega_3^2 = -\frac{1}{2}(S_1 + S_2) + \frac{1}{3}A_2 - \frac{\sqrt{3}}{2}i(S_1 - S_2) \quad .$$

Once the frequencies have been evaluated, the phase velocities can easily be determined. The displacements may be calculated from the expressions

$$\frac{V}{U}=\frac{n}{\sigma ka}\frac{\Omega^2-\frac{1}{2}(1-\sigma)(2+\sigma)(ka)^2-\frac{1}{2}(1-\sigma)n^2}{\Omega^2-\frac{1}{2}(1-\sigma)(ka)^2+\frac{1-\sigma}{2\sigma}n^2}$$

$$\frac{W}{U}=i\frac{\Omega^2-(ka)^2-\frac{1}{2}(1-\sigma)n^2}{\sigma ka}-i\frac{1+\sigma}{2\sigma}n\frac{V}{U}$$

along with the normalization $|U|^2+|V|^2+|W|^2=1$.

Let us consider some examples. We will again let $\sigma=0.3$ and $\beta=0.01$. It is useful to review the $n=0$ case. Figure 5 shows the frequency roots, while figure 6 shows the corresponding phase velocities. Note the crossover of roots 2 and 3. The displacement amplitudes for the three roots are shown in figures 7 through 9. The tangential displacement does not appear because it is decoupled from the others. The behavior of the $n=1$ case is shown in figures 10 through 14, while that of the $n=2$ case is shown in figures 15 through 19.

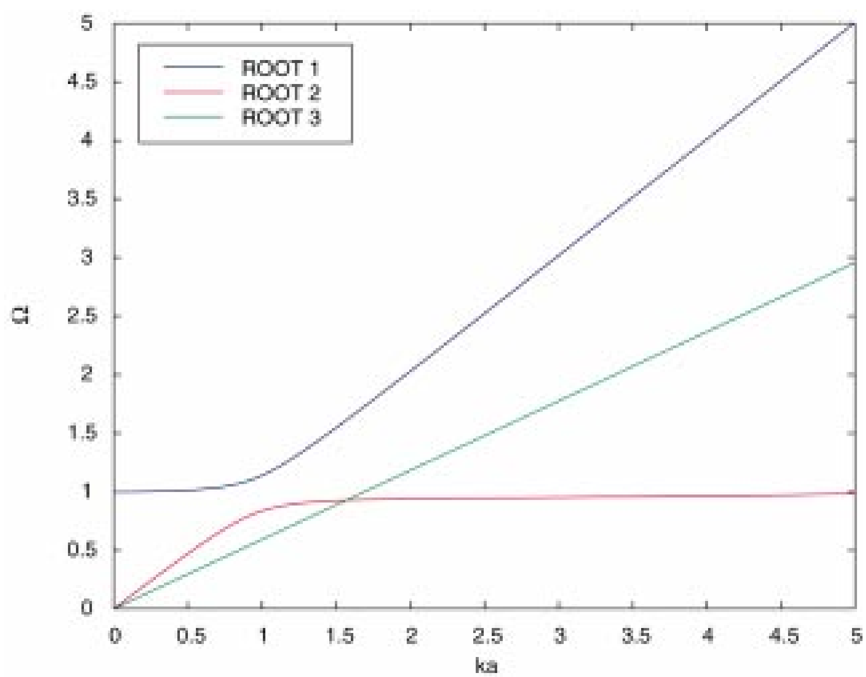

Figure 5. Frequencies for the n = 0 case.

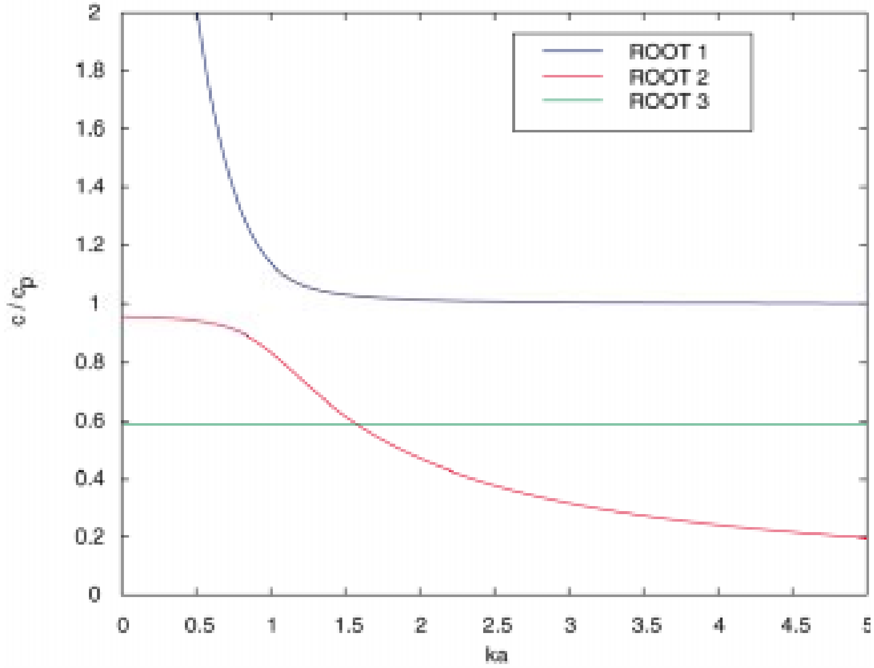

Figure 6. Phase velocities for the n = 0 case.

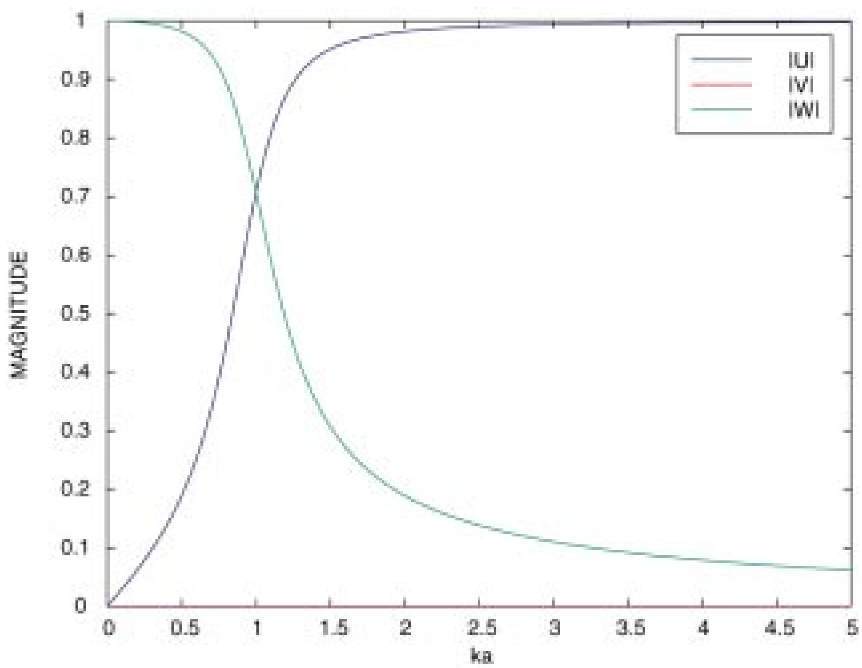

Figure 7. Displacements for root 1 of the n = 0 case.

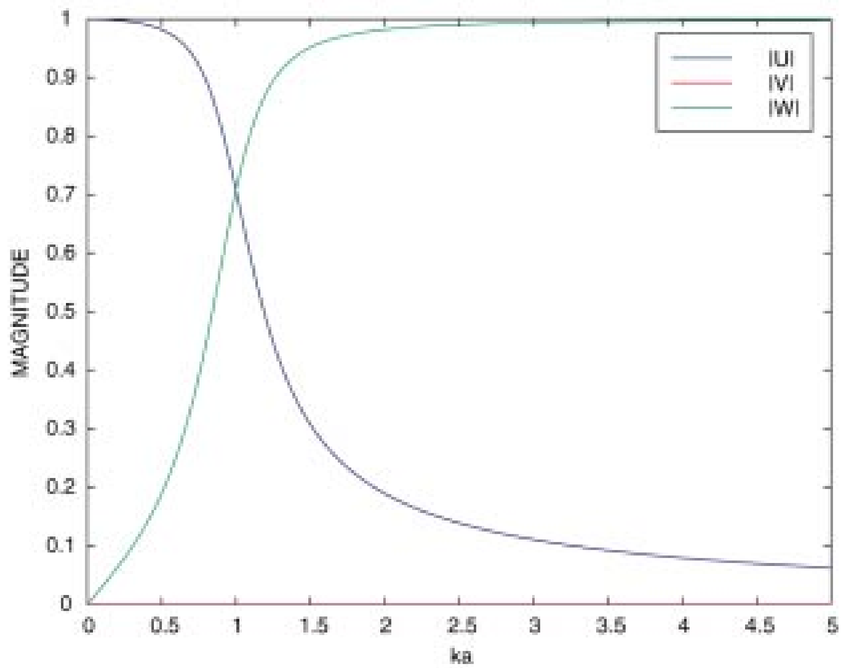

Figure 8. Displacements for root 2 of the n = 0 case.

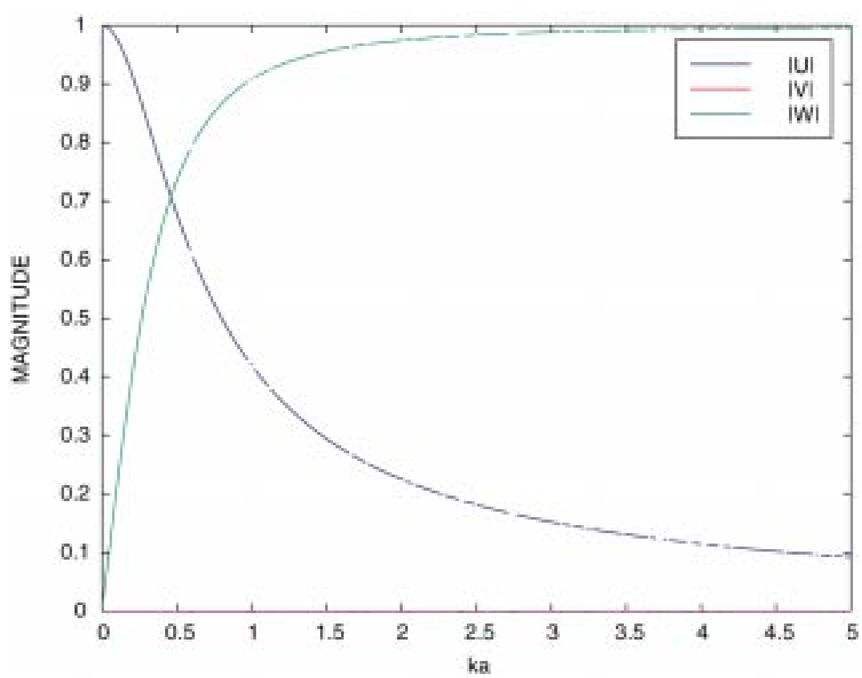

Figure 9. Displacements for root 3 of the n = 0 case.

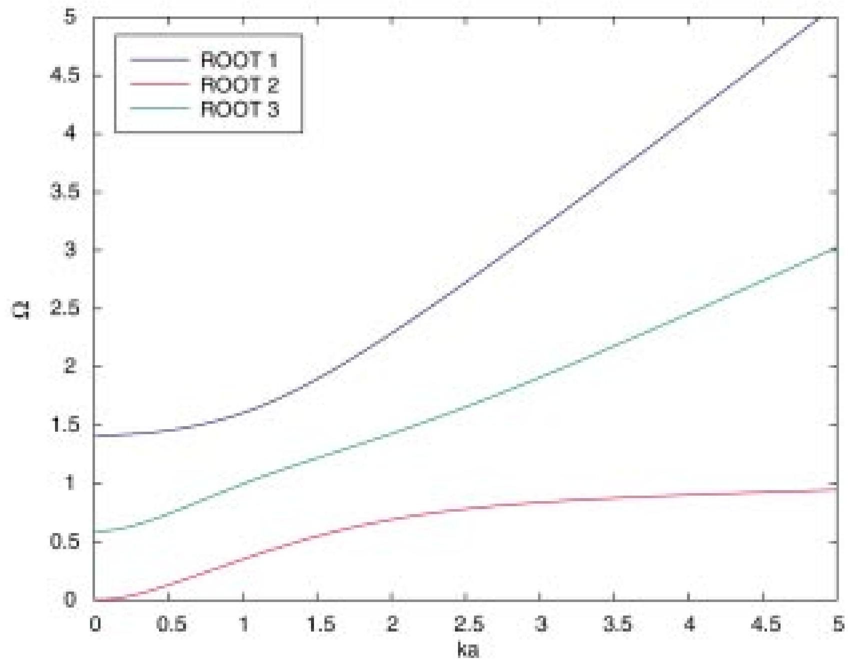

Figure 10. Frequencies for the n = 1 case.

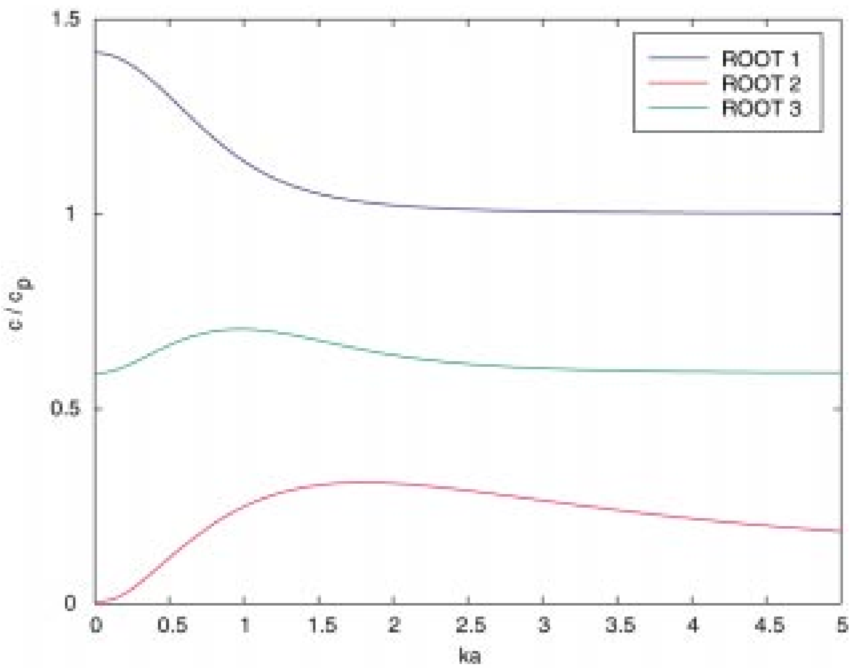

Figure 11. Phase velocities for the n = 1 case.

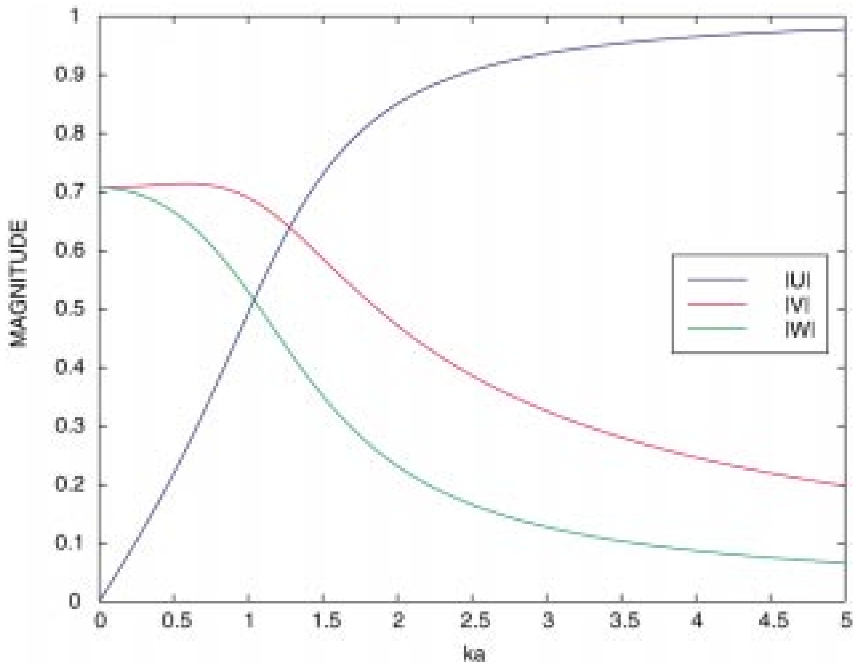

Figure 12. Displacements for root 1 of the n = 1 case.

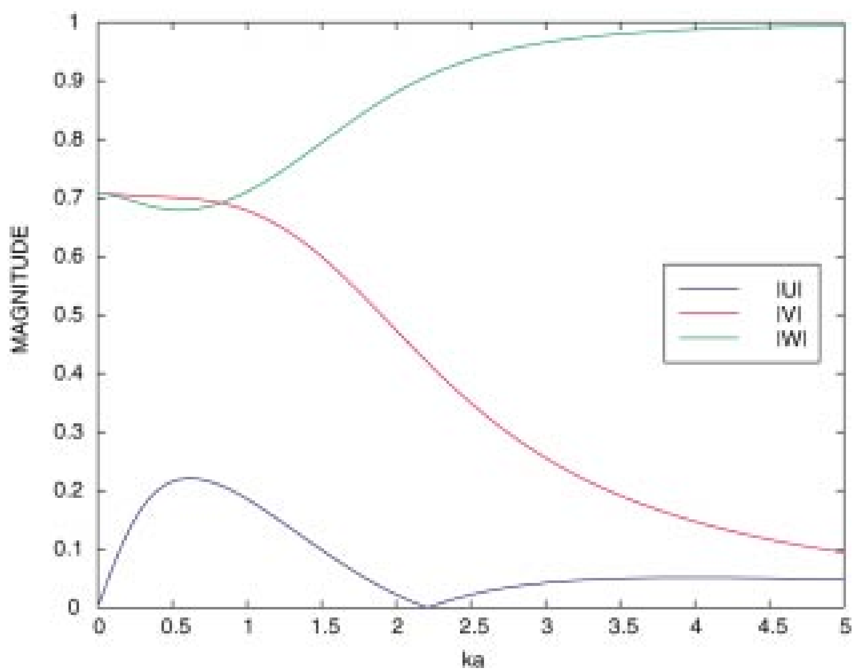

Figure 13. Displacements for root 2 of the n = 1 case.

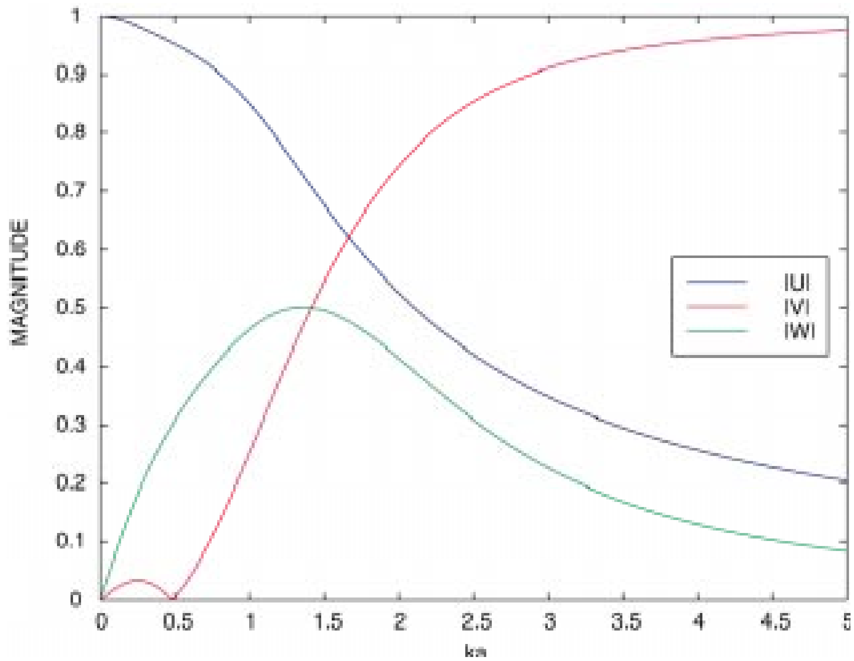

Figure 14. Displacements for root 3 of the n = 1 case.

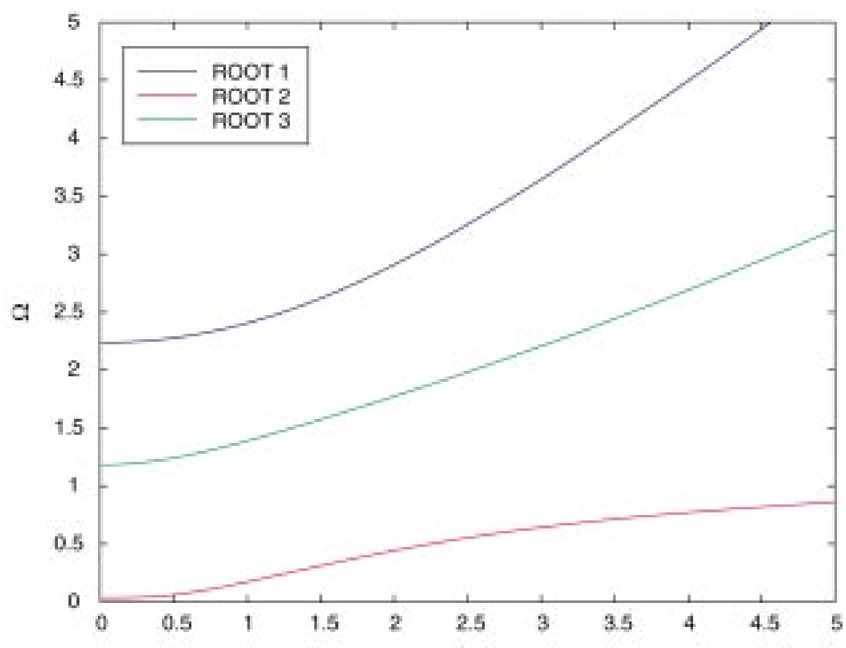

Figure 15. Frequencies for the n = 2 case.

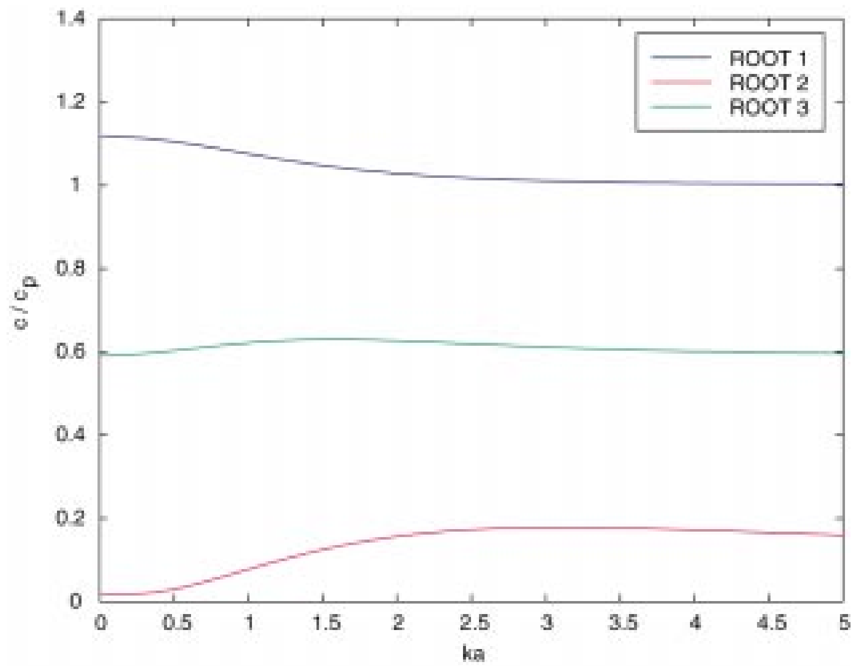

Figure 16. Phase velocities for the n = 2 case.

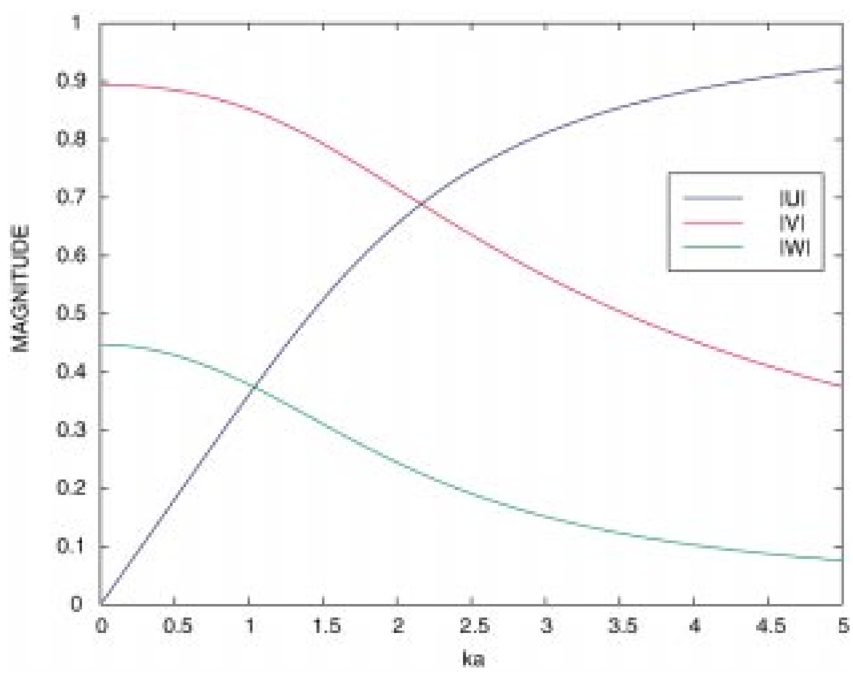

Figure 17. Displacements for root 1 of the n = 2 case.

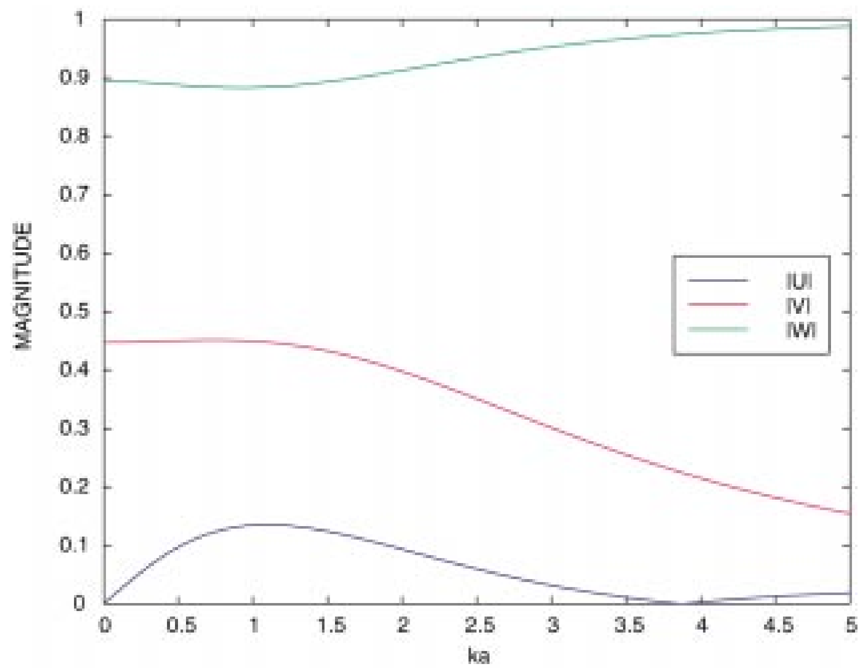

Figure 18. Displacements for root 2 of the n = 2 case.

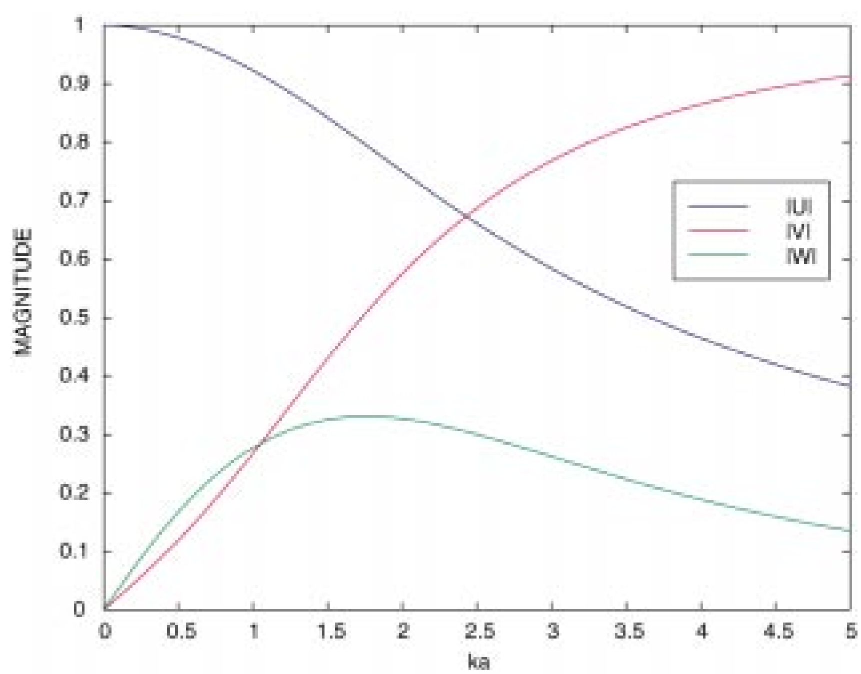

Figure 19. Displacements for root 3 of the n = 2 case.

Forced Vibrations When β = 0

Let us consider forced vibrations from an applied load, which again is symmetric in θ but now has a harmonic variation in z

$$q = f(\theta)e^{i(kz-\omega t)}$$

with

$$f(\theta) = \sum_{n=0}^{\infty} f_n \cos(n\theta).$$

For simplicity, we will assume $\beta = 0$.

Let

$$u = \sum_{n=0}^{\infty} U_n \cos(n\theta)e^{i(kz-\omega t)}$$

$$v = \sum_{n=1}^{\infty} V_n \sin(n\theta)e^{i(kz-\omega t)}$$

$$w = \sum_{n=0}^{\infty} W_n \cos(n\theta)e^{i(kz-\omega t)}.$$

The equations of motion (after using the trigonometric orthogonality relations) are then

$$\begin{bmatrix} \Omega^2 - (ka)^2 - \frac{1-\sigma}{2}n^2 & i\frac{1+\sigma}{2}nka & i\sigma ka \\ -i\frac{1+\sigma}{2}nka & \Omega^2 - \frac{1-\sigma}{2}(ka)^2 - n^2 & -n \\ -i\sigma ka & -n & \Omega^2 - 1 \end{bmatrix} \begin{bmatrix} U_n \\ V_n \\ W_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{f_n a^2}{\rho h c_p^2} \end{bmatrix}.$$

Therefore,

$$u = i\sigma ka \frac{a^2}{\rho h c_p^2} \sum_{n=0}^{\infty} \frac{\Omega^2 + \frac{1-\sigma}{2\sigma}n^2 - \frac{1-\sigma}{2}(ka)^2}{\Omega^6 - A_2\Omega^4 + A_1\Omega^2 - A_0} f_n \cos(n\theta)e^{i(kz-\omega t)}$$

$$v = -\frac{a^2}{\rho h c_p^2} \sum_{n=1}^{\infty} \frac{\Omega^2 - \frac{1-\sigma}{2}n^2 - \frac{(2+\sigma)(1-\sigma)}{2}(ka)^2}{\Omega^6 - A_2\Omega^4 + A_1\Omega^2 - A_0} n f_n \sin(n\theta)e^{i(kz-\omega t)}$$

$$w = -\frac{a^2}{\rho h c_p^2} \sum_{n=0}^{\infty} \frac{\Omega^4 - \frac{3-\sigma}{2}(k_s a)^2\Omega^2 + \frac{1-\sigma}{2}(k_s a)^4}{\Omega^6 - A_2\Omega^4 + A_1\Omega^2 - A_0} f_n \cos(n\theta)e^{i(kz-\omega t)}.$$

Here,

$$A_2 = 1 + \frac{3-\sigma}{2}(k_s a)^2$$

$$A_1 = \frac{1-\sigma}{2}\left[(k_s a)^4 + (k_s a)^2 + 2(1+\sigma)(ka)^2\right]$$

$$A_0 = \frac{1-\sigma}{2}\left(1-\sigma^2\right)(ka)^4 .$$

It may be easily verified that, as $k \to 0$, $u \to 0$ and our earlier results for v and w are recovered.

Thus, the modal mechanical impedance is

$$Z_n^s(\Omega, ka) = -i\frac{\rho c_p}{\Omega}\frac{h}{a}\frac{\Omega^6 - A_2\Omega^4 + A_1\Omega^2 - A_0}{\left[\Omega^2 - \frac{1-\sigma}{2}(k_s a)^2\right]\left[\Omega^2 - (k_s a)^2\right]}$$

and

$$\frac{\partial u}{\partial t} = -i\sigma ka\sum_{n=0}^{\infty}\frac{\Omega^2 - \frac{1-\sigma}{2}(k_s a)^2 + \frac{1-\sigma^2}{2\sigma}n^2}{\left[\Omega^2 - \frac{1-\sigma}{2}(k_s a)^2\right]\left[\Omega^2 - (k_s a)^2\right]}\frac{f_n\cos(n\theta)e^{i(kz-\omega t)}}{Z_n^s}$$

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty}\frac{\Omega^2 - \frac{1-\sigma}{2}(k_s a)^2 - \frac{1-\sigma^2}{2}(ka)^2}{\left[\Omega^2 - \frac{1-\sigma}{2}(k_s a)^2\right]\left[\Omega^2 - (k_s a)^2\right]}\frac{nf_n\sin(n\theta)e^{i(kz-\omega t)}}{Z_n^s}$$

$$\frac{\partial w}{\partial t} = \sum_{n=0}^{\infty}\frac{f_n\cos(n\theta)e^{i(kz-\omega t)}}{Z_n^s} .$$

Again, note that these are the *in vacuo* vibrations of the shell.

FINITE-LENGTH SHELLS

For a cylindrical shell of finite length, the specification of boundary conditions is needed. The simplest set is that of a simply supported shell. If the ends are located at $z = \pm\frac{1}{2}L$, the displacement boundary conditions are (Junger and Feit, 1986, p. 218)

$$w = \frac{\partial^2 w}{\partial z^2} = v = \frac{\partial u}{\partial z} = 0 \quad \text{for} \quad z = \pm\frac{1}{2}L .$$

The motion of the shell may be described by

$$u = \sum_{m,n} U_{mn} \cos(n\theta)\sin(k_m z)e^{-i\omega t}$$

$$v = \sum_{m,n} V_{mn} \sin(n\theta)\cos(k_m z)e^{-i\omega t}$$

$$w = \sum_{m,n} W_{mn} \cos(n\theta)\cos(k_m z)e^{-i\omega t}$$

where $k_m = (2m+1)\frac{\pi}{L}$. A solution with the $\cos(n\theta)$ and $\sin(n\theta)$ factors interchanged would also be valid.

For reference, the boundary conditions for a shell that is clamped at its ends are

$$u = v = w = \frac{\partial w}{\partial z} = 0 \quad \text{for} \quad z = \pm\frac{1}{2}L .$$

RADIATION LOADING

A submerged shell undergoing vibrations will radiate into the surrounding fluid. As usual, we will assume a time dependence $e^{-i\omega t}$. Let $P(r,\theta,z)$ denote the radiated pressure field, and let ρ_0 be the density of the fluid and c_0 be the wave speed in the fluid. In the fluid, P must satisfy the Helmholtz equation

$$\nabla^2 P + k^2 P = 0$$

where $k = \frac{\omega}{c_0}$. In cylindrical coordinates, this is

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right]P + k^2 P = 0 .$$

The standard solution is to use separation of variables. Let $P(r,\theta,z) = R(r)\Theta(\theta)Z(z)$. Then $Z = e^{\pm i k_m z}$, $\Theta = e^{\pm i n\theta}$, and

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}+\left(k^2-k_m^2\right)\right]R=0\,.$$

Therefore, $R=AJ_n\left(\sqrt{k^2-k_m^2}\;r\right)+BY_n\left(\sqrt{k^2-k_m^2}\;r\right)$, where $J_n(x)$ and $Y_n(x)$ are Bessel and Weber functions (Gautschi, 1965, p. 358).

The boundary condition relating P to the shell motion is

$$\left.\frac{\partial P}{\partial r}\right|_{r=a}=-\rho_0\frac{\partial^2 w}{\partial t^2}\,.$$

This represents continuity of the normal component of the force at the boundary. Since a fluid does not support shear motion, the tangential components of the force may be discontinuous.

The radial displacement will be written as in the previous section. Since we desire a solution in the form of an outgoing wave, the pressure field is expressible as

$$P(r,\theta,z,t)=\sum_{m,n}A_{mn}H_n^{(1)}\left(\sqrt{k^2-k_m^2}\;r\right)\cos(n\theta)\cos(k_m z)e^{-i\omega t}$$

where $H_n^{(1)}(x)$ is the Hankel function of the first kind (Gautschi, 1965, p. 358). A_{mn} is determined from the radial boundary condition

$$A_{mn}=-\frac{\rho_0\ddot{W}_{mn}}{\sqrt{k^2-k_m^2}\,H_n'^{(1)}\left(\sqrt{k^2-k_m^2}\,a\right)}$$

where $\ddot{W}_{mn}=-\omega^2 W_{mn}$, and the prime denotes differentiation with respect to the argument.

Note that the surface pressure obtained from these equations may be written as

$$P(a,\theta,z,t)=\sum_{m,n}Z_{mn}^r\dot{W}_{mn}\cos(n\theta)\cos(k_m z)e^{-i\omega t}$$

where

$$Z_{mn}^r=\frac{i\rho_0 c_0 k}{\sqrt{k^2-k_m^2}}\frac{H_n^{(1)}\left(\sqrt{k^2-k_m^2}\,a\right)}{H_n'^{(1)}\left(\sqrt{k^2-k_m^2}\,a\right)}$$

is the modal radiation impedance. Thus, no energy is radiated when $k<k_m$.

The Hankel functions may be approximated by their Debye asymptotic expansions (Gautschi, 1965, p. 366).

$$H_\nu^{(1,2)}(z)\approx\sqrt{\frac{2}{\pi z\sin\gamma}}\left[1\mp i\frac{1+\frac{5}{3}\cot^2\gamma}{8z\sin\gamma}\right]e^{\pm i\left[z(\sin\gamma-\gamma\cos\gamma)-\frac{\pi}{4}\right]}$$

$$H_\nu'^{(1,2)}(z) \approx \pm i\sqrt{\frac{2\sin\gamma}{\pi z}}\left[1 \pm 3i\frac{1+\frac{7}{9}\cot^2\gamma}{8z\sin\gamma}\right]e^{\pm i\left[z(\sin\gamma-\gamma\cos\gamma)-\frac{\pi}{4}\right]}$$

in which $\gamma = \cos^{-1}\left(\frac{\nu}{z}\right)$ and $0 \le \mathrm{Re}(\gamma) \le \pi$. These expansions are generally valid for $|z| > |\nu|$ and $|z-\nu| > |\nu|^{1/3}$. Note that $\sin\gamma = \frac{\sqrt{z^2-\nu^2}}{z}$. Thus,

$$\frac{H_\nu^{(1,2)}(z)}{H_\nu'^{(1,2)}(z)} \approx \mp iz\left[\frac{1}{\sqrt{z^2-\nu^2}} \mp \frac{1}{2}i\frac{z^2}{(z^2-\nu^2)^2}\right].$$

It is worth commenting that some authors ignore the $\cot^2\gamma$ term in the Debye expansions and consequently have an invalid approximation for the ratio (Rumerman, 1996).

Now, if $\sqrt{k^2-k_m^2}a >> n$,

$$Z_{mn}^r \approx \rho_0 c_0 \frac{k}{\sqrt{k^2-k_m^2}}.$$

The case of z-independent motion may be obtained by simply setting $k_m = 0$. Then,

$$Z_n^r = i\rho_0 c_0 \frac{H_n^{(1)}(ka)}{H_n'^{(1)}(ka)}$$

and, if $ka >> n$,

$$Z_n^r \approx \rho_0 c_0$$

the specific acoustic impedance of the fluid.

The effect of radiation loading on forced vibrations may be easily determined. From the equation for *in vacuo* vibrations (generalized to axial wavenumbers k_m) $\dot{W}_{mn} = f_{mn}/Z_{mn}^s$. In a fluid, f_{mn} is replaced by $f_{mn} - P_{mn}$, where the minus sign accounts for the direction of the force. Now, as discussed earlier, $P_{mn} = Z_{mn}^r \dot{W}_{mn}$ so that

$$\dot{W}_{mn} = \frac{f_{mn} - Z_{mn}^r \dot{W}_{mn}}{Z_{mn}^s}$$

or

$$\dot{W}_{mn} = \frac{f_{mn}}{Z_{mn}^s + Z_{mn}^r}.$$

Thus, the total (or shell-fluid system) modal impedance Z_{mn}^{T} is the sum of the shell and radiation modal impedances, as might have been expected.

SCATTERING OF PLANE WAVES

Normal Incidence

Incident Wave. Let us consider a plane acoustic wave, given by $P_i(x, y, t) = P_0 \exp[i(kx - \omega t)]$, normally incident upon an infinite cylindrical shell in a fluid. In cylindrical coordinates

$$P_i(r, \theta, t) = P_0 e^{i(kr\cos\theta - \omega t)}.$$

Note that $\theta = 0$ corresponds to the forward-scattering direction. We may represent P_i as a Fourier expansion (Gautschi, 1965, p. 361).

$$P_i(r, \theta, t) = P_0 e^{-i\omega t} \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(kr) \cos(n\theta).$$

Note that since $J_{-n}(z) = (-1)^n J_n(z)$, we may write

$$P_i(r, \theta, t) = P_0 e^{-i\omega t} \sum_{n=-\infty}^{\infty} i^n J_n(kr) \cos(n\theta)$$

and eliminate the ε_n factor. It is often useful to use a representation in terms of Hankel functions

$$P_i(r, \theta, t) = \frac{1}{2} P_0 e^{-i\omega t} \sum_{n=-\infty}^{\infty} i^n \cos(n\theta) \left[H_n^{(1)}(kr) + H_n^{(2)}(kr) \right].$$

Rigid Body: Blocked Pressure. In analyses of scattering from elastic shells, it is useful to first consider scattering from a rigid object of the same shape. The scattered pressure field will be denoted P_s^{Rigid}. The total pressure, which in this case is termed the blocked pressure, is then $P_b = P_i + P_s^{\text{Rigid}}$.

Since the body is rigid, the resultant fluid acceleration must have a zero component along the normal to the boundary

$$\left. \frac{\partial^2 w_s^{\text{Rigid}}}{\partial \mathrm{t}^2} \right|_{r=a} + \left. \frac{\partial^2 w_i}{\partial \mathrm{t}^2} \right|_{r=a} = 0 .$$

$\dfrac{\partial^2 w_i}{\partial \mathrm{t}^2}$ is the normal fluid acceleration that would be observed on the surface in the absence of the scatterer, that is, when the pressure field is identical to the incident pressure. This is given by

$$\rho_0 \left. \frac{\partial^2 w_i}{\partial \mathrm{t}^2} \right|_{r=a} = - \left. \frac{\partial P_i}{\partial r} \right|_{r=a} .$$

Combining these equations gives

$$\rho_0 \frac{\partial^2 w_s^{\text{Rigid}}}{\partial t^2}\bigg|_{r=a} = \frac{\partial P_i}{\partial r}\bigg|_{r=a}.$$

The corresponding surface acceleration distribution is

$$\frac{\partial^2}{\partial t^2} w_s^{\text{Rigid}}(\theta) = P_0 \frac{k}{\rho_0} \sum_{n=0}^{\infty} \varepsilon_n i^n J_n'(ka)\cos(n\theta).$$

So,

$$\ddot{W}_n = P_0 \frac{k}{\rho_0} \varepsilon_n i^n J_n'(ka).$$

Substituting these coefficients into the expressions from the Radiation Loading section gives

$$P_s^{\text{Rigid}}(r,\theta,t) = -P_0 e^{-i\omega t} \sum_{n=0}^{\infty} \varepsilon_n i^n \frac{J_n'(ka)}{H_n'^{(1)}(ka)} H_n^{(1)}(kr)\cos(n\theta).$$

Thus, the blocked pressure is

$$P_b = P_0 e^{-i\omega t} \sum_{n=0}^{\infty} \varepsilon_n i^n \left[J_n(kr) - \frac{J_n'(ka)}{H_n'^{(1)}(ka)} H_n^{(1)}(kr) \right] \cos(n\theta)$$

or alternatively

$$P_b = \frac{1}{2} P_0 e^{-i\omega t} \sum_{n=-\infty}^{\infty} i^n \left[H_n^{(2)}(kr) - \frac{H_n'^{(2)}(ka)}{H_n'^{(1)}(ka)} H_n^{(1)}(kr) \right] \cos(n\theta).$$

The subscript 1 corresponds to outgoing waves, and the subscript 2 corresponds to incoming waves.

Elastic Shell. While a rigid scatterer distorts the pressure field by interfering with the propagation of the incident wave, the dynamic response of an elastic scatterer further modifies the pressure field. The pressure scattered by an elastic body will be denoted P_s^{Elast}. The total pressure is then $P_T = P_i + P_s^{\text{Elast}}$.

It is convenient to express P_s^{Elast} as the sum of P_s^{Rigid} and an unknown component P_r (Junger and Feit, 1986, p. 343)

$$P_s^{\text{Elast}} \equiv P_s^{\text{Rigid}} + P_r.$$

To interpret P_r, note that the boundary condition (on radial motion) is satisfied if

$$\frac{\partial P_r}{\partial r}\bigg|_{r=a} = -\rho_0 \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial P_i}{\partial r}\bigg|_{r=a} = -\frac{\partial P_s^{\text{Rigid}}}{\partial r}\bigg|_{r=a}.$$

The latter condition is automatically fulfilled through the definition of P_s^{Rigid}. The former condition indicates that P_r is equivalent to the radiated pressure field due to the acceleration of the elastic body responding to the blocked pressure field.

We will use the Hankel function representation of the blocked pressure to allow comparison with the results of Rumerman (1991). We therefore seek a solution in the form

$$P_r(r,\theta,t) = \sum_{n=-\infty}^{\infty} A_n H_n^{(1)}(kr)\cos(n\theta)e^{-i\omega t}.$$

The A_n are determined from the boundary condition

$$A_n = -\frac{\rho_0 \ddot{W}_n}{kH_n'^{(1)}(ka)}$$

or equivalently

$$A_n = \frac{i\rho_0 c_0 \dot{W}_n}{H_n'^{(1)}(ka)}.$$

Now

$$\dot{W}_n = -\frac{P_{bn}}{Z_n^s + Z_n^r}$$

where the minus sign accounts for the direction of the force, and again Z_n^s and Z_n^r are the modal shell and radiation impedances. Consequently,

$$A_n = -\frac{1}{2}P_0 \frac{i\rho_0 c_0}{Z_n^s + Z_n^r}\frac{i^n}{H_n'^{(1)}(ka)}\left[H_n^{(2)}(ka) - \frac{H_n'^{(2)}(ka)}{H_n'^{(1)}(ka)}H_n^{(1)}(ka)\right].$$

Thus, the total pressure field may be represented by the normal mode series

$$P_T(r,\theta,t) = \frac{1}{2}P_0 e^{-i\omega t}\sum_{n=-\infty}^{\infty} i^n \cos(n\theta)\left[H_n^{(2)}(kr) - H_n^{(1)}(kr)R_n(\Omega,ka)\frac{H_n'^{(2)}(ka)}{H_n'^{(1)}(ka)}\right]$$

where

$$R_n(\Omega,ka) = \frac{Z_n^s(\Omega,ka) + i\rho_0 c_0 \dfrac{H_n^{(2)}(ka)}{H_n'^{(2)}(ka)}}{Z_n^s(\Omega,ka) + i\rho_0 c_0 \dfrac{H_n^{(1)}(ka)}{H_n'^{(1)}(ka)}}.$$

R_n is a generalization of the plane-wave reflection coefficient, to which it reduces when $ka >> 1$ and $ka >> n$.

The denominator of R_n is the modal impedance of the shell-fluid system

$$Z_T(\Omega, ka) = Z_n^s(\Omega, ka) + i\rho_0 c_0 \frac{H_n^{(1)}(ka)}{H_n'^{(1)}(ka)}.$$

Note that the shell modal impedance may be written as

$$Z_n^s(\Omega, ka) = -i\rho c_p \frac{h}{a}\Omega\left[1 - \frac{1}{\Omega^2 - n^2} - \frac{\beta^2 n^4}{\Omega^2}\right].$$

Here, $\Omega = \frac{c_0}{c_p} ka$.

Rumerman (1991) expresses the total pressure field as a contour integral with n generalized to a complex number ν. This is noteworthy because the poles of the integrand are the zeroes of the system impedance, viewed as a function of ν. Since Z_ν^T is an even function of ν, the poles appear in equal and opposite pairs. There is a quasi-real pole with $\mathrm{Re}(\nu) \approx \Omega$ corresponding to a quasi-compressional wave. Another quasi-real pole with $\mathrm{Re}(\nu) \approx \sqrt{\Omega/\beta}$ corresponds to a quasi-flexural wave. These are generalizations of our previous results, modified by the effect of radiation loading. However, here there is an infinity of poles ν_{Creep} corresponding to creeping waves. There are also poles corresponding to the exponentially decaying quasi-flexural near-field.

Rumerman shows that, when $|\nu_{\mathrm{Flex}}| << ka$, the flexural wave is very poorly coupled to the fluid and makes an insignificant contribution to the scattered field. The creeping waves, though important in forward scattering, decay so rapidly as they circumnavigate the shell that their backscattering contributions are also small. Thus, if we restrict ourselves to the backscattering half-space $|\theta| > \pi/2$, these terms may be neglected. An additional simplification is that quite often $\beta^2 << 1$, and when $|\nu| \le ka$, quantities relating to Z_n^s may be accurately determined with the bending term ignored. Therefore, we may take

$$Z_\nu^s(\Omega, ka) \approx -i\rho c_p \frac{h}{a}\Omega\left[1 - \frac{1}{\Omega^2 - n^2}\right].$$

Now, in the backscattering half-space, for most or all of that space (depending on the value of ka), the Hankel functions may be approximated by their Debye asymptotic expansions. Then,

$$R_\nu(\Omega, ka) \approx \frac{\left[Z_\nu^s(\Omega, ka) - \frac{i\rho_0 c_0}{2ka\sin^4\gamma}\right] - \frac{\rho_0 c_0}{\sin\gamma}}{\left[Z_\nu^s(\Omega, ka) - \frac{i\rho_0 c_0}{2ka\sin^4\gamma}\right] + \frac{\rho_0 c_0}{\sin\gamma}}$$

and

$$Z_\nu^T(\Omega, ka) \approx -i\rho c_p \frac{h}{a}\Omega\left[1 - \frac{1}{\Omega^2 - n^2}\right] - i\frac{\rho_0 c_0}{2ka\sin^4\gamma} + \frac{\rho_0 c_0}{\sin\gamma}$$

where $\sin\gamma = \dfrac{\sqrt{(ka)^2 - \nu^2}}{ka}$.

The system pole ν_{Comp} is found by setting $Z_\nu^T = 0$. An approximate, though very accurate, solution is found by letting $\sin\gamma = 1$

$$\nu_{\text{Comp}}^2 \approx \Omega^2 - \left[1 + \frac{i}{ka}\frac{\rho_0}{\rho}\frac{a}{h}\left(1 - \frac{i}{2ka}\right)\right]^{-1}.$$

We, of course, are interested in values of ka where ν_{Comp} is quasi-real and an integer. That is, we want $\nu_{\text{Comp}} = n(1 + i\delta)$ with $|\delta| << 1$. Here,

$$n^2 \approx \left(\frac{c_0}{c_p}\right)^2 (ka)^2 - \frac{(ka)^2 + \frac{1}{2}\frac{\rho_0}{\rho}\frac{a}{h}}{(ka)^2 + \left(\frac{\rho_0}{\rho}\frac{a}{h}\right)^2}$$

$$\delta \approx \frac{1}{2}\frac{ka}{n^2}\frac{\frac{\rho_0}{\rho}\frac{a}{h}}{(ka)^2 + \left(\frac{\rho_0}{\rho}\frac{a}{h}\right)^2}.$$

Note that these conditions cannot be satisfied by $n = 0$. The $n = 0$ mode does not couple to the fluid and may be ignored (Rumerman, 1993). More fundamentally, an imaginary part of the root is needed to couple to the fluid (Rumerman, 1992). For typical values of the parameters, the condition $|\delta| << 1$ is satisfied if $ka > 3/2$. A check shows that $\sin\gamma = \sqrt{1 - (c_0/c_p)^2} \approx 1$, as was earlier assumed.

Oblique Incidence

Let us now consider a plane acoustic wave obliquely incident upon an infinite cylindrical shell in a fluid

$$P_i(r, \theta, z, t) = P_0 \exp[i(kr\cos\theta\cos\phi + kz\sin\phi - \omega t)]$$

where ϕ is the angle of incidence (with respect to the normal). This problem is formally identical to that of normal incidence through the replacement of k by $K = k\cos\phi$ (or of the sound speed in the fluid c_0 by $c_0/\cos\phi$). However, the resulting behavior has significant differences (Rumerman, 1992). These are that an obliquely incident wave may excite two kinds of membrane modes on the shell, while only one is excited at normal incidence, and that each of these modes is a supersonic wave only within a range of angles about normal (and evanescent elsewhere).

The incident pressure may be expressed as

$$P_i(r,\theta,z,t)=\frac{1}{2}P_0 e^{i(k_m z-\omega t)}\sum_{n=-\infty}^{\infty} i^n\cos(n\theta)\left[H_n^{(1)}(Kr)+H_n^{(2)}(Kr)\right]$$

so that the blocked pressure is

$$P_b=\frac{1}{2}P_0 e^{i(k_m z-\omega t)}\sum_{n=-\infty}^{\infty} i^n\cos(n\theta)\left[H_n^{(2)}(Kr)-\frac{H_n'^{(2)}(Ka)}{H_n'^{(1)}(Ka)}H_n^{(1)}(Kr)\right]$$

where we have defined an axial wavenumber $k_m = k\sin\phi$.

In a manner analogous to that used in the previous section, the total pressure may be represented by the normal mode series

$$P_T(r,\theta,z,t)=\frac{1}{2}P_0 e^{i(k_m z-\omega t)}\sum_{n=-\infty}^{\infty} i^n\cos(n\theta)\left[H_n^{(2)}(Kr)-H_n^{(1)}(Kr)R_n(\Omega,ka,\phi)\frac{H_n'^{(2)}(Ka)}{H_n'^{(1)}(Ka)}\right]$$

where

$$R_n(\Omega,ka,\phi)=\frac{Z_n^s(\Omega,k_m a)+i\dfrac{\rho_0 c_0}{\cos\phi}\dfrac{H_n^{(2)}(Ka)}{H_n'^{(2)}(Ka)}}{Z_n^s(\Omega,k_m a)+i\dfrac{\rho_0 c_0}{\cos\phi}\dfrac{H_n^{(1)}(Ka)}{H_n'^{(1)}(Ka)}}.$$

The modal system impedance is

$$Z_T(\Omega,ka,k_m a)=Z_n^s(\Omega,k_m a)+i\frac{\rho_0 c_0}{\cos\phi}\frac{H_n^{(1)}(Ka)}{H_n'^{(1)}(Ka)}.$$

The total pressure may be expressed as a contour integral (Rumerman, 1993). As before, the poles appear in equal and opposite pairs. Here too, when $|\nu_{\mathrm{Flex}}| << ka$, the flexural wave is very poorly coupled to the fluid. If we again restrict ourselves to the backscattering half-space $|\theta| > \pi/2$, the creeping wave contribution may be neglected.

We will again ignore the bending terms in the modal shell impedance (Rumerman, 1993) which may be written as

$$Z_\nu^s(\Omega,k_m a)=-i\left[\frac{\rho c_p}{\Omega}\frac{h}{a}\right]\frac{A_R(k_m a)^4+B_R(k_m a)^2+C_R}{D_R(k_m a)^4+E_R(k_m a)^2+F_R}$$

with

$$A_R=\frac{1-\sigma}{2}\left[\Omega^2-(1-\sigma^2)\right]$$

$$B_R=(1-\sigma)\nu^2\Omega^2-\frac{1-\sigma}{2}\Omega^2\left[\Omega^2-1\right]-\Omega^2\left[\Omega^2-(1-\sigma^2)\right]$$

$$C_R = \Omega^2\left[\Omega^2 - \frac{1-\sigma}{2}\nu^2\right]\left[\Omega^2 - 1 - \nu^2\right]$$

$$D_R = \frac{1-\sigma}{2}$$

$$E_R = (1-\sigma)\nu^2 - \frac{3-\sigma}{2}\Omega^2$$

$$F_R = \left[\Omega^2 - \frac{1-\sigma}{2}\nu^2\right]\left[\Omega^2 - \nu^2\right].$$

Recall that $\Omega = \frac{c_0}{c_p}ka$.

We may use the Debye expansions to approximate the generalized reflection coefficient as

$$R_\nu(\Omega, ka, \phi) \approx \frac{\left[Z_\nu^s(\Omega, k_m a) - \frac{i\rho_0 c_0}{2Ka\sin^4\gamma\cos\phi}\right] - \frac{\rho_0 c_0}{\sin\gamma\cos\phi}}{\left[Z_\nu^s(\Omega, k_m a) - \frac{i\rho_0 c_0}{2Ka\sin^4\gamma\cos\phi}\right] + \frac{\rho_0 c_0}{\sin\gamma\cos\phi}}$$

with $\gamma = \cos^{-1}\left(\frac{\nu}{Ka}\right)$. The modal system impedance is

$$Z_\nu^T(\Omega, ka, k_m a) = Z_\nu^s(\Omega, k_m a) - \frac{i\rho_0 c_0}{2Ka\sin^4\gamma\cos\phi} + \frac{\rho_0 c_0}{\sin\gamma\cos\phi}.$$

Here we will not be able to make the simplifying approximation $\sin\gamma = 1$ to find the poles. Instead, replace ν with $Ka\cos\gamma$, then replace $\cos^2\gamma$ with $1-\sin^2\gamma$, and solve for $\sin\gamma$. This yields six independent solutions, most of which are discarded because they violate the Debye expansion criteria. The two valid solutions are denoted γ_{Comp} and γ_{Shear}. The corresponding poles are $\nu_{\text{Comp}} = Ka\cos\gamma_{\text{Comp}}$ and $\nu_{\text{Shear}} = Ka\cos\gamma_{\text{Shear}}$. Rumerman (1993) discusses the typical behavior of these poles for $ka\cos\phi > 3/2$. When $\phi < \phi_{\text{Comp}} = \sin^{-1}(c_0/c_p)$, ν_{Comp} is essentially real (with a small imaginary part) representing a supersonic wave. When $\phi > \phi_{\text{Comp}}$, ν_{Comp} is essentially imaginary (with a small real part) representing an evanescent mode. Similarly, when $\phi < \phi_{\text{Shear}} = \sin^{-1}(c_0/c_s)$, ν_{Shear} is essentially real representing a supersonic wave, and when $\phi > \phi_{\text{Shear}}$, ν_{Shear} is essentially imaginary representing an evanescent mode.

Since

$$\sin\gamma = \frac{\sqrt{(k^2 - k_m^2)a^2 - \nu^2}}{\sqrt{k^2 - k_m^2}a}$$

$$\cos\phi = \frac{\sqrt{k^2 - k_m^2}}{k},$$

the modal radiation impedance may be expressed as

$$Z_\nu^r(ka, k_m a) \approx \rho_0 c_0 ka \left\{ \frac{1}{\sqrt{(k^2 - k_m^2)a^2 - \nu^2}} - \frac{1}{2} i \frac{(k^2 - k_m^2)a^2}{\left[(k^2 - k_m^2)a^2 - \nu^2\right]^2} \right\}.$$

Thus, the zeroes of the modal system impedance may be found from

$$0 = -i\left[\frac{\rho c_p}{\Omega}\frac{h}{a}\right]\left[(k^2 - k_m^2)a^2 - \nu^2\right]^2 \left[A_R (k_m a)^4 + B_R (k_m a)^2 + C_R\right] +$$
$$\left[\rho_0 c_0 ka\right]\left\{\left[(k^2 - k_m^2)a^2 - \nu^2\right]^{3/2} - \frac{1}{2} i (k^2 - k_m^2)a^2\right\}\left[D_R (k_m a)^4 + E_R (k_m a)^2 + F_R\right].$$

Note that, if $k_m = 0$, this becomes

$$0 = -i\left[\frac{\rho c_p}{\Omega}\frac{h}{a}\right]\left[(ka)^2 - \nu^2\right]^2 \Omega^2 \left[\Omega^2 - 1 - \nu^2\right]\left[\Omega^2 - \frac{1-\sigma}{2}\nu^2\right] +$$
$$\left[\rho_0 c_0 ka\right]\left\{\left[(ka)^2 - \nu^2\right]^{3/2} - \frac{1}{2} i (ka)^2\right\}\left[\Omega^2 - \nu^2\right]\left[\Omega^2 - \frac{1-\sigma}{2}\nu^2\right].$$

Then, $\nu_{\text{Shear}} = \sqrt{\frac{2}{1-\sigma}}\Omega = \frac{c_0}{c_s} ka$ is purely real. The associated membrane mode has purely in-plane motion that does not couple to the fluid (Rumerman, 1992). It may, therefore, be disregarded. Now, $\sqrt{1 - \left(\frac{\nu}{ka}\right)^2} \approx 1$, and the previous result for ν_{Comp} is retrieved.

The equation for the zeroes of the modal system impedance must be solved numerically, and is tedious. Figure 20 shows the first five compressional and shear modes when $a = 3$ m, $h = 0.05$ m, $\rho = 7800$ kg/m^3, $\sigma = 0.3$, $E = 2.04 \times 10^{11}$ N/m^2, $\rho_0 = 1000$ kg/m^3, and $c_0 = 1500$ m/s. Here, $c_p = 5360$ m/s and $c_s = 3170$ m/s, so $\phi_{\text{Comp}} = 16.3°$ and $\phi_{\text{Shear}} = 28.2°$.

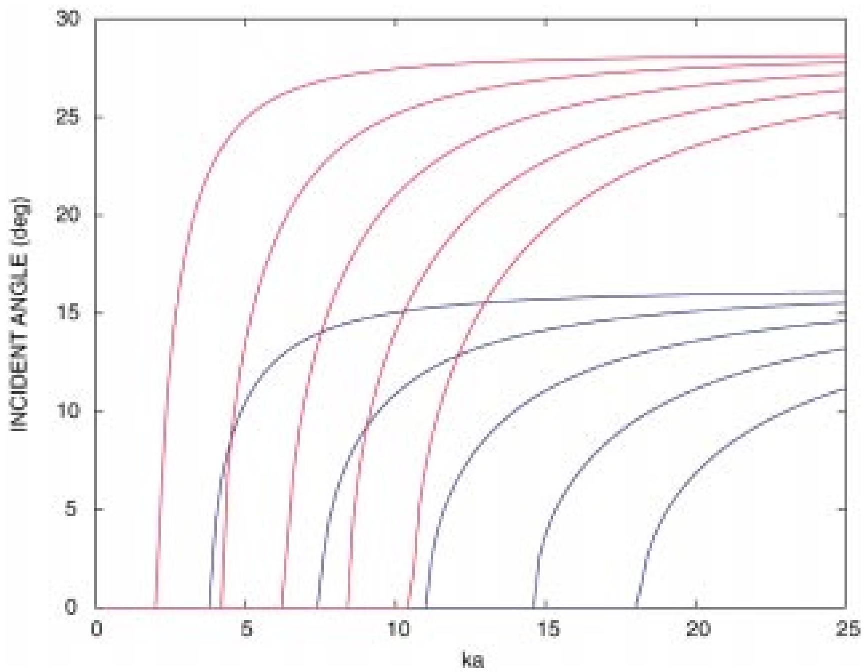

Figure 20. Example of dispersion curves as functions of frequency
and incident angle.

AMPLITUDE

Rumerman (1993) has shown that, by use of a Sommerfeld–Watson type transformation, the scattered field may be expressed in terms of a series of axial modes having coefficients that are waveforms in the circumferential direction. The remainder of the field is taken as a "geometric" contribution.

$$P(R,\theta,\phi) = P^g + P^w$$

where (R,θ,ϕ) are the spherical coordinates of the observation point.

We are interested in the wave component of the scattered field. In the far field, this may be evaluated using the method of stationary phase. For simplicity, the details will be omitted. The backscatter result is

$$P^w(R,\pi,-\phi_{\text{inc}}) = P_0 \frac{\rho\omega a}{K_{\text{inc}} a} \frac{a}{R} e^{ikR} \left[\sum_{m=1}^{M_c} (-i)^m \Gamma_m^{(c)} + \sum_{m=1}^{M_s} (-i)^m \Gamma_m^{(s)} \right]$$

where the parenthetical superscripts distinguish between the compressional and shear contributions, and M_c is the largest integer not exceeding $(1+\Omega)\bar{L}/\pi$ while M_s is the largest integer not exceeding $(1+\omega a/c_s)\bar{L}/\pi$. Here, the normalized length $\bar{L} = L/a$. Recall that $K_{\text{inc}} a = ka \cos\phi_{\text{inc}}$. The parameters associated with the incident field have been explicitly called out for clarity. The coefficients $\Gamma_m^{(j)}$ are given by

$$\Gamma_m^{(j)} = \left[\frac{T_{\beta_m\nu}S_{\alpha_{\text{inc}}\nu} - S_{\beta_m\nu}T_{\alpha_{\text{inc}}\nu}}{T_{\beta_m\nu}S_{\alpha_m\nu} - S_{\beta_m\nu}T_{\alpha_m\nu}}\right]\left[\frac{e^{i\alpha_{\text{inc}}\bar{L}} - e^{i\alpha_m\bar{L}}}{Z_\nu^T(\Omega,\alpha_{\text{inc}})}\right]\left[\frac{1-e^{-im\pi+i\alpha_{\text{inc}}\bar{L}}}{m\pi - \alpha_{\text{inc}}\bar{L}}\right]\left[\frac{1}{\sin(\nu\pi)\dfrac{d\alpha_\nu}{d\nu}}\right]\times$$
$$\left[\frac{e^{-2iK_{\text{inc}}a(\sin\delta_m - \delta_m\cos\delta_m)}}{\sin\delta_m}\right].$$

Here, α_m and β_m are the two linearly independent nondimensional axial wavenumbers in the membrane range (for brevity, the superscripts j have been omitted), $\alpha_{\text{inc}} = ka\sin\phi_{\text{inc}}$, and $\cos\delta_m = \nu/(K_{\text{inc}}a)$. The parameters S and T are normalized displacements and stresses. Their detailed expressions are not needed in what is to follow.

A particular mode will contribute strongly to backscattering if two conditions are satisfied. The first is that $\text{Re}(\nu_m)$ is approximately equal to an integer and $\text{Im}(\nu_m) << 1$. This condition is independent of the angle ϕ_{inc} and signifies the mode is close to resonance at the frequency considered. The second condition is that $\alpha_{\text{inc}} = ka\sin\phi_{\text{inc}}$ is approximately equal to α_m. This indicates the axial variation of the projection of the incident plane wave on the shell is close to the axial variation of the mode shape.

Because $Z_\nu^T(\Omega,\alpha_m) \equiv 0$ for $\nu = \nu_m$, $Z_\nu^T(\Omega,\alpha_{\text{inc}}) \approx (\alpha_{\text{inc}} - \alpha_m)Z_\nu'^T(\Omega,\alpha_m)$ when $\alpha_{\text{inc}} \approx \alpha_m$, where the prime denotes differentiation with respect to α_m. In addition, $Z_\nu'^T(\Omega,\alpha_m)(d\alpha_m/d\nu) = -\dot{Z}_\nu^T(\Omega,\alpha_m)$ where the dot denotes differentiation with respect to ν. Therefore when $\alpha_{\text{inc}} \approx \alpha_m$

$$\Gamma_m^{(j)} \approx \left[\frac{T_{\beta_m\nu}S_{\alpha_{\text{inc}}\nu} - S_{\beta_m\nu}T_{\alpha_{\text{inc}}\nu}}{T_{\beta_m\nu}S_{\alpha_m\nu} - S_{\beta_m\nu}T_{\alpha_m\nu}}\right]\left[\frac{1-e^{-im\pi+i\alpha_{\text{inc}}\bar{L}}}{m\pi - \alpha_{\text{inc}}\bar{L}}\right]^2\left[\frac{(-1)^m\bar{L}}{\sin(\nu\pi)\dot{Z}_\nu^T(\Omega,\alpha_m)}\right]\times$$
$$\left[\frac{e^{-2iK_{\text{inc}}a(\sin\delta_m - \delta_m\cos\delta_m)}}{\sin\delta_m}\right].$$

The quantity in the second set of brackets is a directivity function of that mode. If the angular width of the mode is less than the angular distance to the adjacent modes, the peak level of a resonant mode can be estimated ignoring other contributions. Rumerman uses the following relationships. When $\alpha_{\text{inc}} = \alpha_m$, the quantity in each of the first two sets of brackets is equal to unity. For $\text{Re}(\nu_m)$ an integer n,

$$\sin(\pi\nu_m) = i(-1)^n\sinh[\pi\,\text{Im}(\nu_m)] \approx i(-1)^n\pi\,\text{Im}(\nu_m).$$

The wavenumber pair (α_m,ν_m) is a joint solution of $0 = Z_\nu^T(\Omega,\alpha_{\text{inc}}) = Z_\nu^s(\Omega,\alpha_m) - iX_\nu^r + R_\nu^r$, where X_ν^r is the reactive component of the radiation impedance and R_ν^r is the resistive component. The shell impedance has been taken to be completely reactive. At high frequencies, where the shell impedance generally dominates the radiation impedance, ν_m may be approximated by $\nu_R + i\nu_I$, where ν_R is the real root of $0 = Z_\nu^s(\Omega, \alpha_m) - iX_\nu^r$. The imaginary component may be found from

$$0 = Z_\nu^T(\Omega, \alpha_m) \approx Z_{\nu_R}^T(\Omega, \alpha_m) + i\nu_I \dot{Z}_{\nu_R}^T(\Omega, \alpha_m) = R_{\nu_R}^r + i\nu_I \dot{Z}_{\nu_R}^T(\Omega, \alpha_m).$$

Therefore, $\nu_I = i R_{\nu_R}^r \big/ \dot{Z}_{\nu_R}^T$, where $R_{\nu_R}^r = \rho\omega a \big/ (K_{\text{inc}} a \sin\delta)$ with $\cos\delta = \nu_R \big/ (K_{\text{inc}} a)$ and $\delta = \delta_m$ for the mode considered.

Upon combining these results, the pressure amplitude backscattered by an isolated resonant mode into the angle at which it is traced-matched with the incident wave may be found. It is simply

$$\left| P^w(R, \pi, -\phi_{\text{inc}}) \right| = P_0 \frac{a}{R} \frac{\overline{L}}{\pi}.$$

Obviously, this expression is not valid for arbitrarily long shells. A limiting value, independent of length, should be reached as $\overline{L} \to \infty$. This limit can be determined from the transformed expression by noting that it represents the scattered field as a series of axial modes. When $\overline{L}$ is very large, the modal density is very high, and no one mode has any significance. The solution must be obtained by summing over all modes. As $\overline{L} \to \infty$, $m\pi / \overline{L}$ may be thought of as a continuous variable and the summation approximated as an integral over this variable. The integral itself can be approximated by noting that the integrand has singularities at values of the integration variable at which ν_m is an integer n. The magnitude of the backscattered pressure due to the circumferential mode of order n, and for $\alpha_{\text{inc}} = \text{Re}(\alpha_n)$, is approximately (Rumerman, 1993).

$$\left| P_{\max}^w(R, \pi, -\phi_{\text{inc}}) \right| = P_0 \frac{a}{R} \frac{2}{\pi \left| \text{Im}(\alpha_n) \right|}.$$

MODELING ISSUES

APPROXIMATE SOLUTIONS OF THE DISPERSION EQUATION

Formulation

Recalling our results from the section "Forced Vibrations When $\beta = 0$," the system impedance in that case is

$$Z_{mn}^{T} = -i\frac{\rho c_p}{\Omega}\frac{h}{a}\frac{\Omega^6 - A_2\Omega^4 + A_1\Omega^2 - A_0}{\left[\Omega^2 - \frac{1-\sigma}{2}(k_s a)^2\right]\left[\Omega^2 - (k_s a)^2\right]} + Z_{mn}^{r}.$$

The dispersion equation $Z_{mn}^{T} = 0$ may be written as

$$0 = \left[\Omega^2 - (k_s a)^2\right]\left[\Omega^2 - \frac{1-\sigma}{2}(k_s a)^2\right]\left[\Omega^2 + i\frac{\Omega a}{\rho c_p h}Z_{mn}^{r}\right]$$
$$-\Omega^4 + \frac{1-\sigma}{2}\left[\Omega^2 (k_s a)^2 + 2(1+\sigma)(k_m a)^2 - (1-\sigma^2)(k_m a)^4\right]$$

where again $k_s a \equiv \sqrt{(k_m a)^2 + n^2}$ and $k_m \equiv k \sin\phi$.

When $\beta \neq 0$, the dispersion equation has a very similar form (Guo, 1994)

$$0 = \left[\Omega^2 - (k_s a)^2\right]\left[\Omega^2 - \frac{1-\sigma}{2}(k_s a)^2\right]\left[\Omega^2 - \beta^2 (k_s a)^4 + i\frac{\Omega a}{\rho c_p h}Z_{mn}^{r}\right]$$
$$-\Omega^4 + \frac{1-\sigma}{2}\left[\Omega^2 (k_s a)^2 + 2(1+\sigma)(k_m a)^2 - (1-\sigma^2)(k_m a)^4\right].$$

Note that in the high-frequency limit, the roots of this equation are given by the three bracketed terms on the first line.

We will follow Guo's approach (Guo, 1994) in taking these high-frequency limits as leading order terms, and seek corrections to extend the solutions into the mid-frequency region. Let

$$\xi = \frac{(k_s a)^2}{\Omega^2} = \frac{(k_m a)^2 + n^2}{\Omega^2}$$

$$\Lambda = \frac{n^2}{\Omega^2}.$$

The dispersion equation then takes the form

$$D_0(\xi)+\frac{1}{\Omega^2}D_1(\xi)=0$$

where ξ is the quantity to be determined and the functions D_0 and D_1 are given by

$$D_0(\xi)=[\xi-1]\left[\frac{1-\sigma}{2}\xi-1\right][\mu\xi^2-1-i\alpha z_{mn}]$$

$$D_1(\xi)=1-\frac{1-\sigma}{2}\Lambda-\frac{1-\sigma}{2}(\xi-\Lambda)[2\sigma+3-(1-\sigma^2)(\xi-\Lambda)]$$

with

$$\mu\equiv\beta^2\Omega^2$$

$$\alpha\equiv\frac{\rho_0 a}{\rho h}$$

$$z_{mn}\equiv\frac{1}{\rho_0 c_0}\frac{Z^r_{mn}}{ka}.$$

It may be seen that the problem has been contrived in such a way as to suggest solutions of the form

$$\xi=\xi_0+\frac{1}{\Omega^2}\xi_1+\cdots$$

Although this takes the nominal form of an inverse power series in frequency, the final results will not have such a form because the frequency parameter also appears in D_0 through both μ and z_{mn}. These will be taken as fixed in applying the expansion. As Guo points out, a strict inverse power series in frequency is not a suitable solution for cases of fluid loading because the roots are generally complex with real parts many orders of magnitude larger than the imaginary parts. An inverse power series expansion would require a large number of terms in order to obtain the first non-zero imaginary part.

In finding the roots corresponding to shear and compressional waves, μ and z_{mn} are assumed to be unaffected by the expansion. The reason for keeping μ constant is purely for convenience. It is always much smaller than unity, under the constraint of thin-shell theory, and consequently, makes a negligible contribution to the roots. The reason for keeping z_{mn} constant is subtler. It is because z_{mn} is the term accounting for fluid (radiation) loading, and hence, the only term that gives the roots imaginary parts. Keeping z_{mn} in D_0 will allow us to find the complex roots with only one correction term. This does not contradict the use of the expansion above, because z_{mn} is of order $1/\Omega$, as may be seen from its definition and our previous analysis. Therefore, if an expansion is begun at the order $1/\Omega^2$, z_{mn} should be included in the leading-order expression.

For flexural waves, these arguments do not hold. Here, the dominant effect of fluid loading is added mass with little radiation. z_{mn} contributes predominantly to the real part of the roots. In fact, in the frequency range of interest, the roots for flexural waves are purely real. Consequently, expansion of z_{mn} in terms of frequency is required. Since flexural waves make a negligible contribution to the scattering problem, we will not pursue their study.

To continue the analysis, we substitute the expansion into the recast dispersion equation, and group terms of like powers (after expanding D_0 in a Taylor series). To order $1/\Omega^2$, this leads to solutions of the form

$$\xi = \xi_0 - \frac{1}{\Omega^2}\frac{D_1(\xi_0)}{D_0'(\xi_0)}$$

where the prime denotes differentiation with respect to the argument and ξ_0 are the high-frequency limits. It may be seen that

$$D_0'(\xi_0) = 2\mu\xi_0[\xi_0 - 1]\left[\frac{1-\sigma}{2}\xi_0 - 1\right] + [\mu\xi_0^2 - 1 - i\alpha z_{mn}]\left[\frac{1-\sigma}{2}\xi_0 - 1\right] + \frac{1-\sigma}{2}[\mu\xi_0^2 - 1 - i\alpha z_{mn}][\xi_0 - 1]$$

and

$$z_{mn}(\xi_0) = \frac{1}{\Omega\sqrt{\left(\frac{c_p}{c_0}\right)^2 - \xi_0}} - \frac{1}{2}i\frac{\Omega^2\left[\left(\frac{c_p}{c_0}\right)^2 - \xi_0\right] + n^2}{\Omega^4\left[\left(\frac{c_p}{c_0}\right)^2 - \xi_0\right]^2}.$$

Note that this differs from Guo's expression.

Compressional Waves

The leading-order solution for compressional waves is

$$\xi_0 = 1.$$

Here,

$$D_0'(\xi_0) = -\frac{1+\sigma}{2}[\mu - 1 - i\alpha z_{mn}]$$

$$D_1(\xi_0) = \frac{1+\sigma}{2}[\sigma + (1-\sigma)\Lambda]^2$$

so that

$$\xi = 1 + \frac{1}{\Omega^2}\frac{[\sigma + (1-\sigma)\Lambda]^2}{\mu - 1 - i\alpha z_{mn}} .$$

In terms of the axial wavenumber, this may be rewritten as

$$(k_m a)^2 = \Omega^2 \left\{ 1 - \frac{n^2}{\Omega^2} + \frac{1}{\Omega^2} \frac{\left[\sigma + (1-\sigma)\frac{n^2}{\Omega^2}\right]^2}{\mu - 1 - i\alpha z_{mn}} \right\}$$

with

$$z_{mn} = \frac{1}{\Omega\sqrt{\left(\frac{c_p}{c_0}\right)^2 - 1}} - \frac{1}{2} i \frac{\Omega^2\left[\left(\frac{c_p}{c_0}\right)^2 - 1\right] + n^2}{\Omega^4\left[\left(\frac{c_p}{c_0}\right)^2 - 1\right]^2} .$$

Figure 21 shows a comparison with the results of Rumerman's formulation. The parameters are the same as before. As can be seen, the differences are minor and are so for most practical problems.

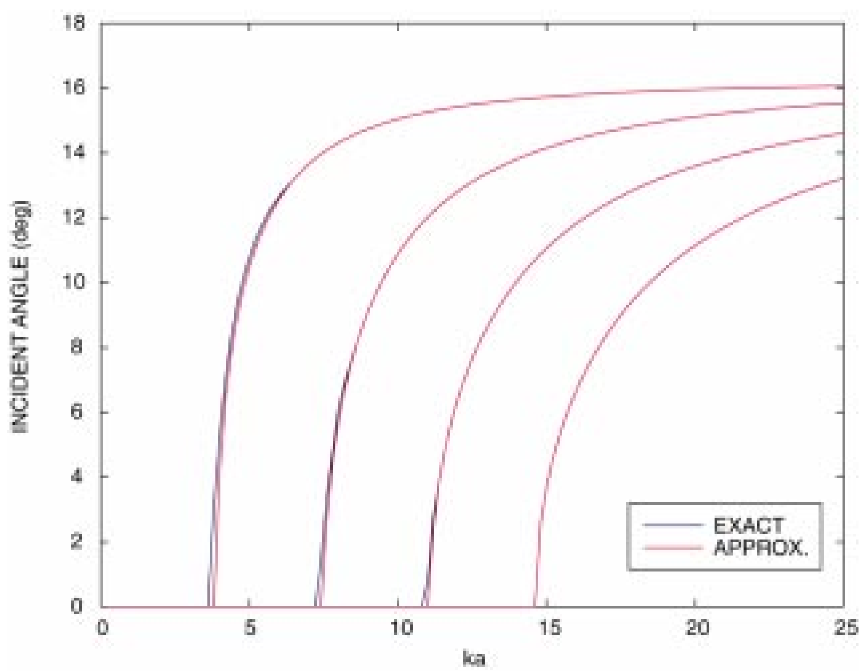

Figure 21. Comparison of approximate and "exact" solution for compressional modes.

Shear Waves

The leading-order solution for shear waves is

$$\xi_0 = \frac{2}{1-\sigma} .$$

Here,

$$D_0'(\xi_0) = 2\frac{1+\sigma}{(1-\sigma)^2}\left[\mu - \frac{1}{4}(1-\sigma)^2 - \frac{1}{4}i(1-\sigma)^2 \alpha z_{mn}\right]$$

$$D_1(\xi_0) = -(1-\sigma^2)\Lambda\left[1-\frac{1}{2}(1-\sigma)\Lambda\right]$$

so that

$$\xi = \frac{2}{1-\sigma} + \frac{1}{\Omega^2}\frac{\frac{1}{2}(1-\sigma)^3\Lambda\left[1-\frac{1}{2}(1-\sigma)\Lambda\right]}{\mu - \frac{1}{4}(1-\sigma)^2 - \frac{1}{4}i(1-\sigma)^2\alpha z_{mn}} .$$

This may be rewritten as

$$(k_m a)^2 = \frac{2\Omega^2}{1-\sigma}\left[1-\frac{1}{2}(1-\sigma)\frac{n^2}{\Omega^2}\right]\left\{1+\frac{1}{\Omega^2}\frac{\frac{1}{4}(1-\sigma)^4\frac{n^2}{\Omega^2}}{\mu - \frac{1}{4}(1-\sigma)^2\left[1+i\alpha z_{mn}\right]}\right\}$$

with

$$z_{mn} = \frac{1}{\Omega\frac{c_p}{c_s}\sqrt{\left(\frac{c_s}{c_0}\right)^2 - 1}} - \frac{1}{2}i\frac{\Omega^2\left(\frac{c_p}{c_s}\right)^2\left[\left(\frac{c_s}{c_0}\right)^2 - 1\right] + n^2}{\Omega^4\left(\frac{c_p}{c_s}\right)^4\left[\left(\frac{c_s}{c_0}\right)^2 - 1\right]^2} .$$

Note that $k_m = 0$ when

$$\Omega = \sqrt{\frac{1-\sigma}{2}}n .$$

This is the cutoff frequency for shear waves. In fact, it is the exact cutoff condition that can be derived directly from the dispersion equation.

Figure 22 shows a comparison with the results of Rumerman's formulation. The differences are insignificant.

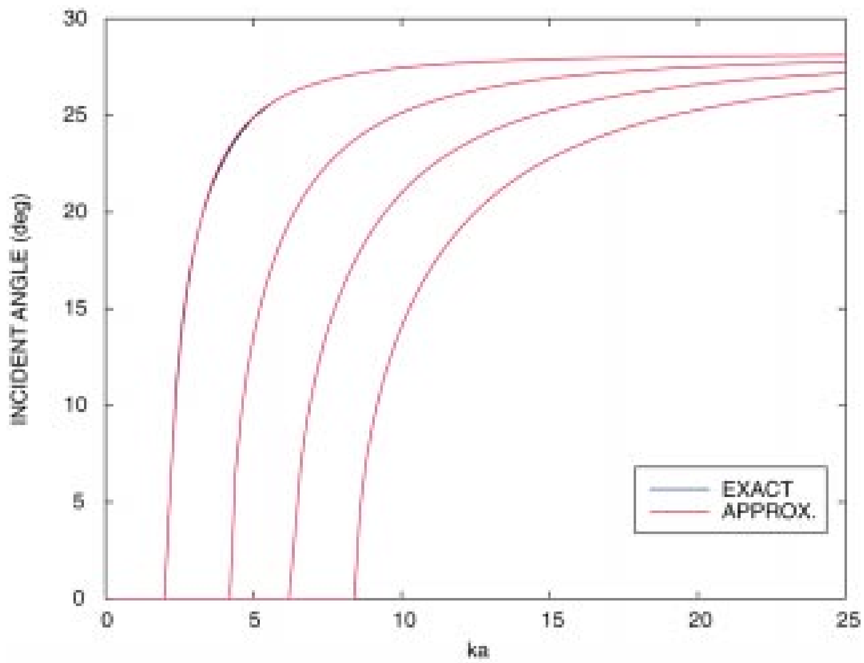

Figure 22. Comparison of approximate and "exact" solution for shear modes.

MODAL WIDTHS

Previously, we had noted the axial modal directivity function

$$B(\chi) \equiv \left|\frac{1-e^{-i\chi}}{\chi}\right|^2 = \left|\frac{\sin\left(\frac{1}{2}\chi\right)}{\frac{1}{2}\chi}\right|^2$$

where

$$\chi = m\pi - ka\sin\phi_{\text{inc}}\bar{L} \quad .$$

As defined, B is an even function in χ and $B(0)=1$. Note that $B=\sqrt{2}/2$ when $\Delta\chi = 2.004$. The axial modal half-width is therefore $\Delta(ka\sin\phi_{\text{inc}}) = \Delta\chi/\bar{L}$. We will later see that taking $\Delta\chi = 1.00$ gives better agreement with observations.

The circumferential modal directivities may be characterized in terms of parameters

$$\zeta_s = \Omega_s\sqrt{1-\frac{n^2}{\Omega_s^2}} - ka\sin\phi_{\text{inc}}$$

$$\zeta_c = \Omega_c\sqrt{1-\frac{n^2+\sigma^2}{\Omega_c^2}} - ka\sin\phi_{\text{inc}} \quad .$$

The circumferential modal half-widths are then $\Delta(ka \sin\phi_{\text{inc}}) = \Delta\zeta$. Here, we do not have an explicit directivity function. For convenience, we will take the directivity to be unity over the modal width and zero otherwise. We will later see that taking $\Delta\zeta = 0.25$ produces behavior in reasonable agreement with observations.

AMPLITUDE

Our previous results for the backscattered pressure amplitude, when traced-matched with the incident wave, may be summarized as

$$\left|P^w(R,\pi,-\phi_{\text{inc}})\right| = P_0 \frac{a}{R}\frac{1}{\pi}\min\left[\overline{L}, \frac{2}{\left|\text{Im}(\alpha_n)\right|}\right] .$$

This obviously ignores the transitional behavior. $\text{Im}(\alpha_n)$ cannot be expressed in a simple manner. However, in the regime of interest, $\text{Im}(\alpha_n) \propto n$ (Rumerman, 1992 and 1993). The ka dependence is less apparent. From our earlier results, we would expect $\text{Im}(\alpha_n) \sim \sqrt{(ka\cos\phi)^2 + (\rho_0 a/\rho h)^2} \big/ ka\cos\phi$. For typical problems, a good fit is

$$\text{Im}(\alpha_n) = 0.366 \frac{\rho}{\rho_0}\frac{h}{a}\frac{\sqrt{(ka\cos\phi)^2 + \left(\frac{\rho_0}{\rho}\frac{a}{h}\right)^2}}{ka\cos\phi} n .$$

It is important to note that, in practice, these peak levels are not observed. There are a number of reasons for this. The primary one has to do with the shell termination. Caps modify the coupling of the incident wave with the shell modes. Stiffeners will as well. Typically, the modal structure is not significantly altered, but the backscattered levels are reduced by some coupling efficiency factor. This factor is usually in the range of 0.6 to 0.8.

EXAMPLE

Let us consider the example used by Rumerman (1993) where $a/h = 100$, $c_p/c_0 = 3.5$, $\rho/\rho_0 = 7.8$, and $\sigma = 0.3$. The model we have presented produces the results shown in figure 23. The color scale represents the target strength in decibels. This may be compared with Rumerman's figure 3 (data) and figure 7 (predictions). The qualitative agreement is very good. Given the uncertainties involved, the quantitative agreement is good. Rumerman explains the reasons for many of the discrepancies. Figure 24 shows the response over a wider bandwidth.

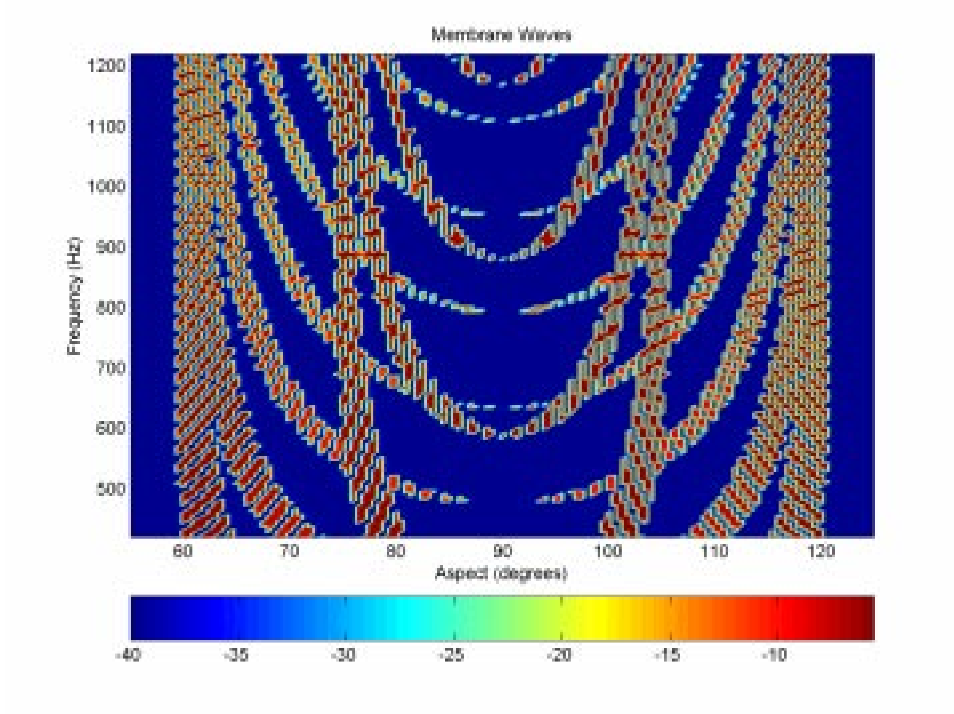

Figure 23. Predicted response for the example given by Rumerman (1993).

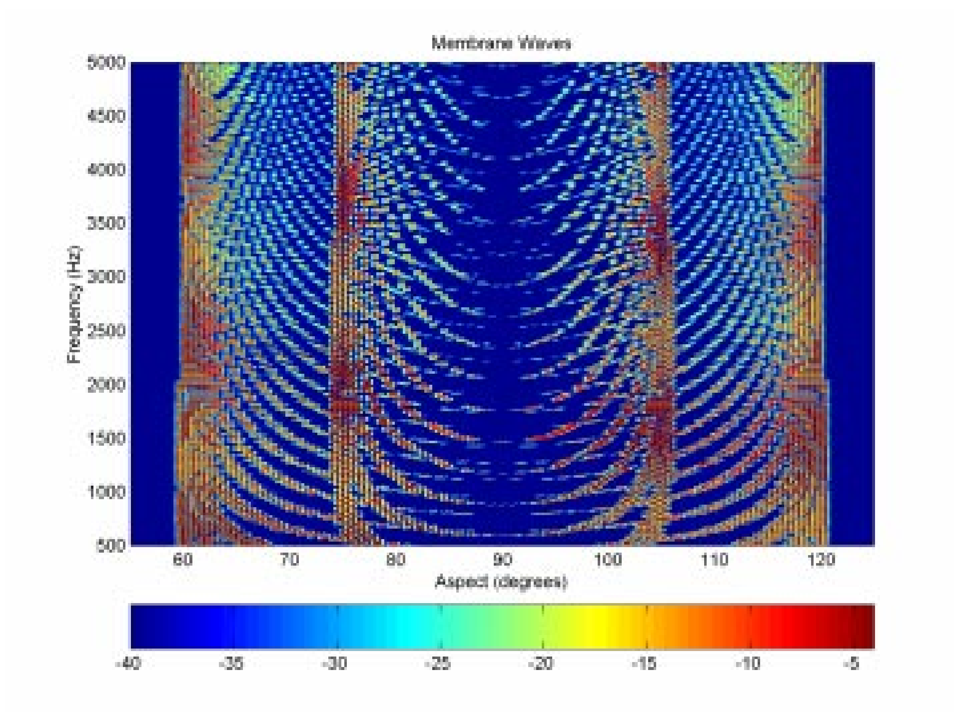

Figure 24. Predicted response over a wider bandwidth.

SUMMARY

We examined the modeling of shell membrane waves and provided detailed theoretical background, both to serve as a primer for those unfamiliar with the subject and to set the context of the model presented. We discussed the nature of the various assumptions and made comparisons of the predicted behavior with “exact” numerical solutions, as well as with experimental data. The qualitative agreement was excellent. Quantitative agreement was good, given the uncertainties involved, as well as sensitivity to boundary conditions.

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This report covers the modeling of shell membrane waves. Detailed theoretical background is provided to serve as a primer on the subject and to set the context. Comparisons are made of the predicted response with “exact” numerical solutions as well as with experimental data. Qualitative agreement is excellent. Quantitative agreement is good, given the uncertainties involved and sensitivity to boundary conditions.

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